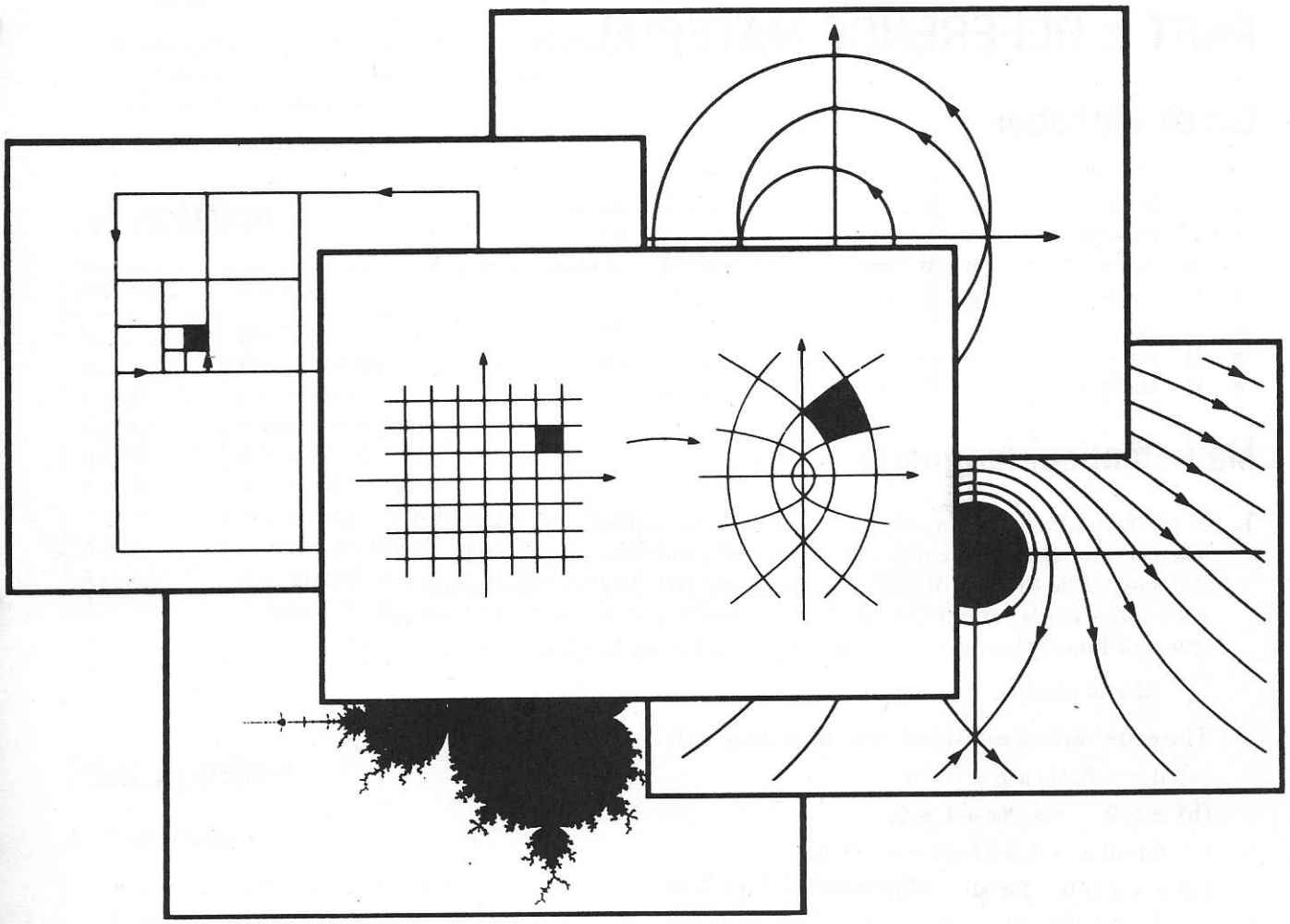
M337 HB1
Mathematics: A Third Level Course

The Open University

COMPLEX ANALYSIS

HANDBOOK ONE



MPLEX ANALYSIS COMPLEX ANALYSIS COMPL

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CONTENTS

Part I: Reference Material 2
- Greek alphabet 2
- Mathematical language 2
- Set notation 3
- Real numbers 3
- Real functions 5

Part II: Unit Summaries 9
- Unit A1 Complex Numbers 9
- Unit A2 Complex Functions 12
- Unit A3 Continuity 15
- Unit A4 Differentiation 18

Part III: Index 20

PART I: REFERENCE MATERIAL

Greek alphabet

α	A	alpha	ι	I	iota	ρ	P	rho
β	B	beta	κ	K	kappa	σ	Σ	sigma
γ	Γ	gamma	λ	Λ	lambda	τ	T	tau
δ	Δ	delta	μ	M	mu	υ	Υ	upsilon
ε	E	epsilon	ν	N	nu	ϕ	Φ	phi
ζ	Z	zeta	ξ	Ξ	xi	χ	X	chi
η	H	eta	o	O	omicron	ψ	Ψ	psi
θ	Θ	theta	π	Π	pi	ω	Ω	omega

Mathematical language

1. In mathematics we commonly use **implications** such as 'if P then Q', where P and Q are statements which can be either true or false. The statement P is the **hypothesis** and the statement Q is the **conclusion**. An implication is **true** if the conclusion is true whenever the hypothesis is true and **false** otherwise. For example, the following implication is true:

if x is positive, then $x+1$ is positive.

There are various equivalent ways of stating such an implication:

(a) if $x > 0$, then $x + 1 > 0$;

(b) $x > 0 \implies x + 1 > 0$;

(c) for all $x > 0$, we have $x + 1 > 0$;

(d) $x + 1 > 0$, for all $x > 0$;

(e) $x + 1 > 0$, whenever $x > 0$;

(f) for $x + 1$ to be positive, it is sufficient that x be positive.

2. The **converse** of an implication is obtained by exchanging the hypothesis and the conclusion. For example, the converse of the (true) implication

if $x > 0$, then $x + 1 > 0$

is

if $x + 1 > 0$, then $x > 0$,

which is false (try $x = 0$).

3. An **equivalence** consists of an implication 'if P then Q' *and* its converse 'if Q then P'. The equivalence is true if both these implications are true. For example, the following equivalence is true:

   $x > 0$ is equivalent to $2x > 0$.

   It could alternatively be stated as follows:

   $x > 0 \iff 2x > 0$;

   $x > 0$ if and only if $2x > 0$;

   $x > 0$ is necessary and sufficient for $2x > 0$.

4. There are three ways of proving implications:
   (a) **direct proof**: we begin by assuming that the hypothesis is true and then argue directly to show that the conclusion is true;
   (b) **proof by contraposition**: we begin by assuming that the conclusion is false and then argue directly to show that the hypothesis is false;
   (c) **proof by contradiction**: we begin by assuming that the hypothesis is true *and* that the conclusion is false, and then argue from both to obtain a contradiction.

   Where possible, types (a) or (b) are preferable, since they establish a direct link between hypothesis and conclusion, but type (c) is often convenient and sometimes essential.

Set notation

Notation	Meaning
$\{x, y, \ldots, z\}$	The set of elements listed in $\{\ldots\}$
$\{x : \ldots\}$	The set of all x such that ... holds
$x \in A$	x belongs to A
$x \notin A$	x does not belong to A
$A \subseteq B$	A is a subset of B: each element of A belongs to B
$A = B$	A is equal to B: $A \subseteq B$ and $B \subseteq A$
$A \subset B$	A is a proper subset of B: $A \subseteq B$ but $A \neq B$
$A \cup B$	A union B: the set of all elements which belong to A or B (or both)
$A \cap B$	A intersection B: the set of all elements which belong to both A and B
$A - B$	A minus B: the set of all elements of A which do not belong to B
$\varnothing$	The empty set

Some texts use $A \subset B$ to mean $A \subseteq B$.

Real numbers

1. Definitions

A **real number** is a number which can be represented by a decimal of the form

$$\pm a_0.a_1a_2a_3\ldots,$$

where a_0 is a non-negative integer and a_1, a_2, a_3, ... are digits. **Rational numbers** (ratios of integers) are represented by recurring decimals and **irrational numbers** are represented by non-recurring decimals. Real numbers are often represented by points on a line, called the **real line**.

2. Some important subsets of the real numbers

Symbol	Subset
$\mathbb{N}$	The set of all natural numbers: $1, 2, 3, \ldots$
$\mathbb{Z}$	The set of all integers: $0, \pm 1, \pm 2, \pm 3, \ldots$
$\mathbb{Q}$	The set of all rational numbers (numbers of the form p/q, where $p, q \in \mathbb{Z}, q \neq 0$)
$\mathbb{R}$	The set of all real numbers
$]a, b[$	$\{x : a < x < b\}$, the open interval from a to b
$[a, b]$	$\{x : a \leq x \leq b\}$, the closed interval from a to b
$]a, b]$	$\{x : a < x \leq b\}$
$[a, \infty[$	$\{x : x \geq a\}$
$]a, \infty[$	$\{x : x > a\}$
$]-\infty, a[$	$\{x : x < a\}$

3. Upper and lower bounds

Suppose that A is a non-empty subset of $\mathbb{R}$. Then A is **bounded above** if there is a real number M such that

$$x \leq M, \quad \text{for all } x \in A.$$

The number M is called an **upper bound** of A. Clearly, any number bigger than M is also an upper bound of A. A **lower bound** of A is defined similarly.

Among all upper bounds of A, the smallest (which always exists if A is bounded above) is called the **least upper bound** of A, or the **supremum** of A, written $\sup A$. The **greatest lower bound**, or the **infimum**, of A, written $\inf A$, is defined similarly.

If A has infinitely many elements, then $\sup A$ and $\inf A$ may or may not belong to A. In contrast, if A has finitely many elements, then $\sup A$ and $\inf A$ are the largest and the smallest elements of A, respectively, and therefore are members of A.

If $\sup A$ belongs to A, then we may use the alternative notation $\max A$ for $\sup A$. Similarly, if $\inf A$ belongs to A, then we may denote it by $\min A$.

Example If $A = [-1, 1[$, then

$$\sup A = 1 \quad \text{and} \quad \inf A = \min A = -1.$$

If $B = \{-1, 0, 1, 2\}$, then

$$\sup B = \max B = 2 \quad \text{and} \quad \inf B = \min B = -1.$$

4. Inequalities

Rules for rearranging inequalities

(a) $a < b \iff b - a > 0.$

(b) $a < b \iff a + c < b + c.$

(c) If $c > 0$, then

$$a < b \iff ac < bc;$$

if $c < 0$, then

$$a < b \iff ac > bc.$$

(d) If $a, b > 0$, then

$$a < b \iff \frac{1}{a} > \frac{1}{b}.$$

(e) If $a, b \geq 0$ and $p > 0$, then

$$a < b \iff a^p < b^p.$$

Corresponding versions of these inequalities exist with *strict* inequalities replaced by *weak* inequalities; that is, with $a < b$ replaced by $a \leq b$.

The **solution set** of an inequality involving an unknown real number x is the set of values of x for which the inequality holds.

Rules for deducing new inequalities from given ones

(a) For all a, b, c in $\mathbb{R}$,

$$a < b \text{ and } b < c \implies a < c. \qquad \text{Transitive Rule}$$

(b) For all a, b, c, d in $\mathbb{R}$, if $a < b$ and $c < d$, then

$$a + c < b + d; \qquad \text{Sum Rule}$$

$$ac < bd \quad \text{(provided that } a, c \geq 0\text{)}. \qquad \text{Product Rule}$$

Modulus

If $x \in \mathbb{R}$, then the **modulus**, or **absolute value**, of x is

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Thus $|x|$ is the distance from the origin to x, and so

(a) $|x| < a \iff -a < x < a$;

(b) $|x| > a \iff x > a$ or $x < -a$;

(c) the distance on the real line from a to b is $|a - b|$.

5. The Principle of Mathematical Induction

Suppose that $P(n)$, $n = 1, 2, 3, \ldots$, is a sequence of propositions such that

(a) $P(1)$ is true, and

(b) whenever $P(k)$ is true, $P(k+1)$ is also true.

Then $P(n)$ is true, for $n = 1, 2, \ldots$.

Example We wish to show that $2^n \geq n$, for $n = 1, 2, \ldots$.

Let $P(n)$ be the proposition $2^n \geq n$.

(a) $P(1)$ is true, since $2^1 \geq 1$.

(b) Suppose that $P(k)$ is true; that is, $2^k \geq k$. Then

$$2^{k+1} = 2 \cdot 2^k \geq 2k = k + k \geq k + 1,$$

so that $P(k+1)$ is true.

Thus $2^n \geq n$, for $n = 1, 2, \ldots$, by the Principle of Mathematical Induction.

Real functions

1. Definitions

A **real function** f is defined by specifying

(a) two subsets A and B of $\mathbb{R}$, and

(b) a rule which associates with each $x \in A$ a unique $y \in B$.

The sets A and B are called the **domain** and the **codomain** of f, respectively. We write

$$f : A \longrightarrow B.$$

If $f : A \longrightarrow B$ and $x \in A$, then $f(x)$ is called the **image of x under f**, or the **value of f at x**. The **image** of the function f is

$$f(A) = \{f(x) : x \in A\}.$$

2. Standard functions

Type	Rule	Domain
Polynomial functions	$a_0 + a_1 x + \cdots + a_n x^n$	$\mathbb{R}$
Rational functions	$p(x)/q(x)$, p and q polynomial functions (q not the zero function)	$\mathbb{R} - \{x : q(x) = 0\}$
Trigonometric functions	$\sin x$ $\cos x$ $\tan x$	$\mathbb{R}$ $\mathbb{R}$ $\mathbb{R} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
Exponential functions	e^x a^x, where $a > 0$	$\mathbb{R}$ $\mathbb{R}$
Natural log function	$\log_e x$	$\{x : x > 0\}$
Hyperbolic functions	$\sinh x$ $\cosh x$ $\tanh x$	$\mathbb{R}$ $\mathbb{R}$ $\mathbb{R}$

Remarks

(a) Definition of e^x:

$$e^x = \sum_{n=0}^{\infty} x^n/n! = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

(b) Definition of $\log_e x$:

$$x = e^y \Longrightarrow y = \log_e x$$

(c) Definition of a^x (for $a > 0$):

$$a^x = \exp(x \log_e a)$$

(d) Index laws:

$$a^{x+y} = a^x a^y, \quad (a^x)^y = a^{xy}, \quad \sqrt[n]{a^m} = a^{m/n}, \quad a^{-x} = 1/a^x$$

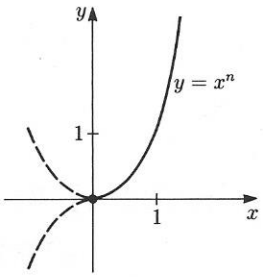
(e) Logarithmic identities:

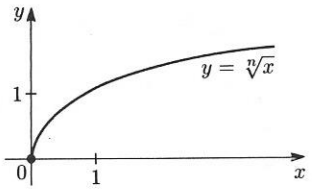
$$\log_e xy = \log_e x + \log_e y, \quad \log_e(1/x) = -\log_e x$$

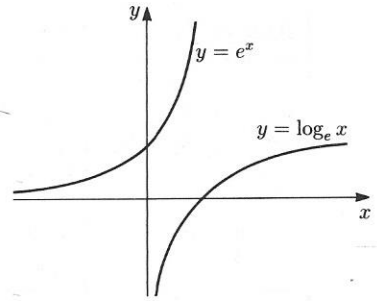
3. Trigonometric and hyperbolic identities

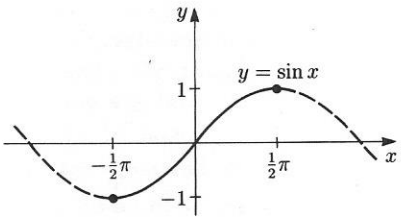
$\cos^2 x + \sin^2 x = 1$ $1 + \tan^2 x = \sec^2 x$ $\cot^2 x + 1 = \operatorname{cosec}^2 x$	$\cosh^2 x - \sinh^2 x = 1$ $1 - \tanh^2 x = \operatorname{sech}^2 x$ $\coth^2 x - 1 = \operatorname{cosech}^2 x$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$ $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
$\sin 2x = 2 \sin x \cos x$ $\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$	$\sinh 2x = 2 \sinh x \cosh x$ $\cosh 2x = \cosh^2 x + \sinh^2 x$ $= 2\cosh^2 x - 1$ $= 1 + 2\sinh^2 x$ $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$
$\sin(-x) = -\sin x$ $\cos(-x) = \cos x$ $\tan(-x) = -\tan x$	$\sinh(-x) = -\sinh x$ $\cosh(-x) = \cosh x$ $\tanh(-x) = -\tanh x$

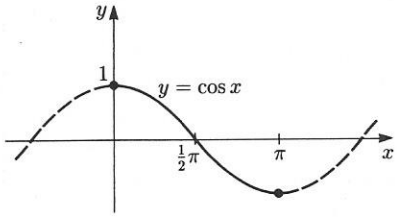
4. Graphs of standard functions and their inverses

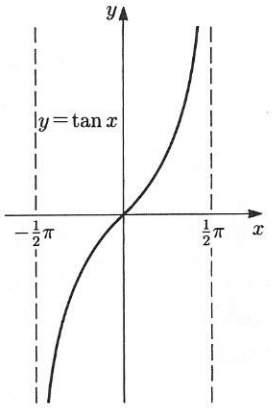


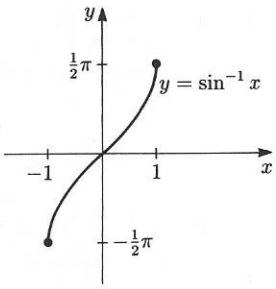


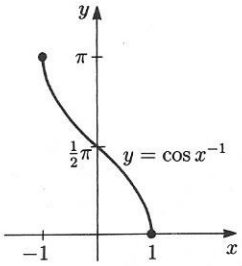


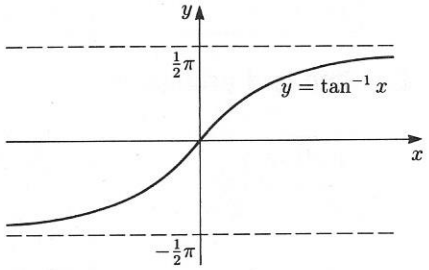


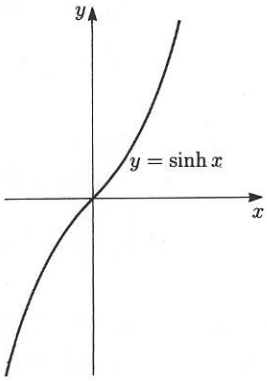


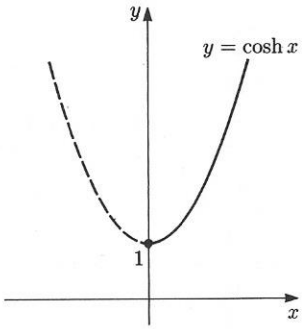


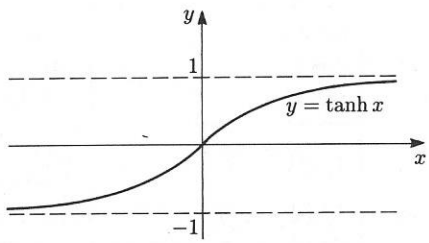


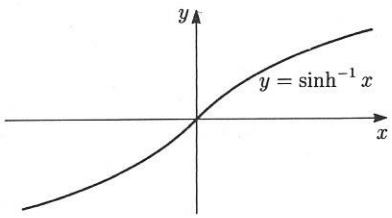


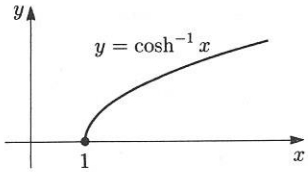


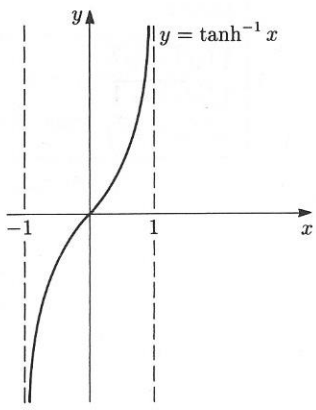












5. Standard derivatives

Rule of f	Rule of f'	Domain of f'
k	0	$\mathbb{R}$
$x^n,\ n \in \mathbb{Z} - \{0\}$	nx^{n-1}	$\mathbb{R}$ if $n > 0$; $\mathbb{R} - \{0\}$ if $n < 0$
$x^\alpha,\ \alpha \in \mathbb{R} - \{0\}$	$\alpha x^{\alpha-1}$	$]0, \infty[$
e^x	e^x	$\mathbb{R}$
$a^x,\ a > 0$	$a^x \log_e a$	$\mathbb{R}$
$\log_e x$	$1/x$	$]0, \infty[$
$\sin x$	$\cos x$	$\mathbb{R}$
$\cos x$	$-\sin x$	$\mathbb{R}$
$\tan x$	$\sec^2 x$	$\mathbb{R} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$
$\sec x$	$\sec x \tan x$	$\mathbb{R} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
$\cot x$	$-\operatorname{cosec}^2 x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$
$\sin^{-1} x$	$1/\sqrt{1-x^2}$	$]-1, 1[$
$\cos^{-1} x$	$-1/\sqrt{1-x^2}$	$]-1, 1[$
$\tan^{-1} x$	$1/(1+x^2)$	$\mathbb{R}$
$\sinh x$	$\cosh x$	$\mathbb{R}$
$\cosh x$	$\sinh x$	$\mathbb{R}$
$\tanh x$	$\operatorname{sech}^2 x$	$\mathbb{R}$
$\sinh^{-1} x$	$1/\sqrt{1+x^2}$	$\mathbb{R}$
$\cosh^{-1} x$	$1/\sqrt{x^2-1}$	$]1, \infty[$
$\tanh^{-1} x$	$1/(1-x^2)$	$]-1, 1[$

6. Standard primitives

Rule of f	Rule of F $(F' = f)$	Domain of F
$x^n,\ n \in \mathbb{Z} - \{-1, 0\}$	$x^{n+1}/(n+1)$	$\mathbb{R}$ if $n > 0$; $\mathbb{R} - \{0\}$ if $n < 1$
$x^\alpha,\ \alpha \in \mathbb{R} - \{-1, 0\}$	$x^{\alpha+1}/(\alpha+1)$	$]0, \infty[$
$1/x$	$\log_e \lvert x \rvert$	$\mathbb{R} - \{0\}$
e^x	e^x	$\mathbb{R}$
$a^x,\ a > 0$	$a^x / \log_e a$	$\mathbb{R}$
$\log_e x$	$x \log_e x - x$	$]0, \infty[$
$\sin x$	$-\cos x$	$\mathbb{R}$
$\cos x$	$\sin x$	$\mathbb{R}$
$\tan x$	$\log_e \lvert \sec x \rvert$	$\mathbb{R} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
$1/\sqrt{1-x^2}$	$\begin{cases} \sin^{-1} x \\ -\cos^{-1} x \end{cases}$	$]-1, 1[$ $]-1, 1[$
$1/(1+x^2)$	$\tan^{-1} x$	$\mathbb{R}$
$\sinh x$	$\cosh x$	$\mathbb{R}$
$\cosh x$	$\sinh x$	$\mathbb{R}$
$\tanh x$	$\log_e(\cosh x)$	$\mathbb{R}$
$1/\sqrt{1+x^2}$	$\sinh^{-1} x$	$\mathbb{R}$
$1/\sqrt{x^2-1}$	$\cosh^{-1} x$	$]1, \infty[$
$1/(1-x^2)$	$\tanh^{-1} x$	$]-1, 1[$

PART II: UNIT SUMMARIES

Unit A1 Complex Numbers

Section 1: Introducing Complex Numbers

1. A **complex number** z is an expression of the form $x + iy$, where $x, y \in \mathbb{R}$ and i is a symbol with the property that $i^2 = -1$. We write $z = x + iy$ or, equivalently, $z = x + yi$, and say that z is expressed in **Cartesian form**; x is called the **real part** of z and y the **imaginary part** of z, written $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

Two complex numbers are **equal** if their real parts are equal and their imaginary parts are equal.

The set of all complex numbers is denoted by $\mathbb{C}$.

2. The binary operations of **addition, subtraction** and **multiplication** of complex numbers are denoted by the same symbols as for real numbers and are performed by treating the complex numbers as real expressions involving an algebraic symbol i with the property that $i^2 = -1$.

3. The **negative**, $-z$, of a complex number $z = x + iy$ is

$$-z = (-x) + i(-y),$$

usually written $-z = -x - iy$.

4. The **reciprocal**, $1/z$ (or z^{-1}), of a non-zero complex number $z = x + iy$ is

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

5. The **quotient**, z_1/z_2, of a complex number z_1 by a non-zero complex number z_2 is

$$\frac{z_1}{z_2} = z_1 \left(\frac{1}{z_2}\right).$$

6. Strategy for obtaining a quotient

To obtain the quotient

$$\frac{x_1 + iy_1}{x_2 + iy_2}, \quad \text{where } x_2 + iy_2 \neq 0,$$

in Cartesian form, multiply both numerator and denominator by $x_2 - iy_2$, so that the denominator becomes real.

7. The **complex conjugate**, $\overline{z}$, of a complex number $z = x + iy$ is

$$\overline{z} = x - iy.$$

8. Properties of the complex conjugate

(a) If z is a complex number, then

(i) $z + \overline{z} = 2 \operatorname{Re} z$;

(ii) $z - \overline{z} = 2i \operatorname{Im} z$;

(iii) $\overline{(\overline{z})} = z$.

(b) If z_1 and z_2 are complex numbers, then

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;

(ii) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$;

(iii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$;

(iv) $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$, where $z_2 \neq 0$.

9. Arithmetic in $\mathbb{C}$

Addition	*Multiplication*
A1 For all z_1, z_2 in $\mathbb{C}$, $z_1 + z_2 \in \mathbb{C}$.	M1 For all z_1, z_2 in $\mathbb{C}$, $z_1 z_2 \in \mathbb{C}$.
A2 For all z in $\mathbb{C}$, $z + 0 = 0 + z = z$.	M2 For all z in $\mathbb{C}$, $z1 = 1z = z$.
A3 For all z in $\mathbb{C}$, $z + (-z) = (-z) + z = 0$.	M3 For all non-zero z in $\mathbb{C}$, $zz^{-1} = z^{-1}z = 1$.
A4 For all z_1, z_2, z_3 in $\mathbb{C}$, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.	M4 For all z_1, z_2, z_3 in $\mathbb{C}$, $(z_1 z_2)z_3 = z_1(z_2 z_3)$.
A5 For all z_1, z_2 in $\mathbb{C}$, $z_1 + z_2 = z_2 + z_1$.	M5 For all z_1, z_2 in $\mathbb{C}$, $z_1 z_2 = z_2 z_1$.
D For all z_1, z_2, z_3 in $\mathbb{C}$, $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.	

10. Binomial Theorem

(a) If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$(1 + z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$$

$$= 1 + nz + \frac{n(n-1)}{2!}z^2 + \cdots + z^n.$$

(b) If $z_1, z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k$$

$$= z_1^n + nz_1^{n-1}z_2 + \frac{n(n-1)}{2!}z_1^{n-2}z_2^2 + \cdots + z_2^n.$$

11. Geometric Series Identity

(a) If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$1 - z^n = (1 - z)\left(1 + z + z^2 + \cdots + z^{n-1}\right).$$

(b) If $z_1, z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$z_1^n - z_2^n$$
$$= (z_1 - z_2)\left(z_1^{n-1} + z_1^{n-2}z_2 + z_1^{n-3}z_2^2 + \cdots + z_2^{n-1}\right).$$

Section 2: The Complex Plane

1. The complex number $z = x + iy$ is represented by the ordered pair (x, y) in $\mathbb{R}^2$, called the **complex plane.**

Complex numbers can be thought of as based vectors; $z = x + iy$ corresponds to the vector from $(0, 0)$ to (x, y). The sum and difference of two complex numbers z_1 and z_2 satisfy the parallelogram law for vectors.

2. The **modulus, or absolute value,** of $z = x + iy$ is the distance from 0 to z; it is denoted by $|z|$. Thus

$$|z| = |x + iy| = \sqrt{x^2 + y^2}.$$

3. $|z_1 - z_2|$ is the distance from z_1 to z_2.
$|z_1 + z_2|$ is the distance from z_1 to $-z_2$.

4. **Properties of the modulus**

(a) $|z| \geq 0$, with equality if and only if $z = 0$;

(b) $|\overline{z}| = |z|$ and $|-z| = |z|$;

(c) $|z|^2 = z\overline{z}$;

(d) $|z_1 - z_2| = |z_2 - z_1|$;

(e) $|z_1 z_2| = |z_1||z_2|$

and

$|z_1/z_2| = |z_1|/|z_2|$, where $z_2 \neq 0$.

5. An **argument** of $z = x + iy$, $z \neq 0$, is an angle θ (measured in radians) such that

$$\cos\theta = \frac{x}{|z|} \quad \text{and} \quad \sin\theta = \frac{y}{|z|}.$$

No argument is assigned to $z = 0$.

Each $z \neq 0$ has infinitely many arguments. If θ is an argument of z, then the arguments of z are given by $\theta + 2n\pi$, $n \in \mathbb{Z}$.

6. The ordered pair (r, θ), where r is the modulus of a non-zero complex number z and θ is an argument of z, are called **polar coordinates** of z. The expression

$$z = r(\cos\theta + i\sin\theta)$$

is called a representation of z in **polar form.**

7. The **principal argument** of a non-zero complex number z is the unique argument θ of z satisfying $-\pi < \theta \leq \pi$; it is denoted by

$$\theta = \operatorname{Arg} z.$$

8. **Strategy for determining principal arguments**

To determine the principal argument θ of a non-zero complex number $z = x + iy$.

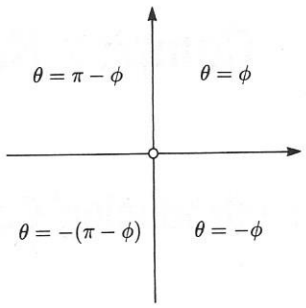
Case (i) If z lies on one of the axes, then θ is evident.

Case (ii) If z does not lie on one of the axes, then

(a) decide in which quadrant z lies (by plotting if necessary) and then calculate the angle

$$\phi = \tan^{-1}(|y|/|x|);$$

(b) obtain θ in terms of ϕ by using the appropriate formula in the following diagram.



9. If z_1 and z_2 are non-zero with

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

The geometric effect on z_1 of multiplying it by z_2 is to scale z_1 by the factor $|z_2|$ and rotate it about 0 through the angle $\operatorname{Arg} z_2$ (this rotation is anticlockwise if $\operatorname{Arg} z_2 > 0$, clockwise if $\operatorname{Arg} z_2 < 0$).

10. If z_1 and z_2 are (non-zero) complex numbers, then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2n\pi,$$

where n is -1, 0 or 1 according as $\operatorname{Arg} z_1 + \operatorname{Arg} z_2$ is greater than π, lies in the interval $]-\pi, \pi]$ or is less than or equal to $-\pi$.

11. If z_1 and z_2 are non-zero with

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).$$

The geometric effect on z_1 of dividing it by z_2 is to scale z_1 by the factor $1/|z_2|$ and rotate it about 0 through the angle $\operatorname{Arg} z_2$ (this rotation is clockwise if $\operatorname{Arg} z_2 > 0$, anticlockwise if $\operatorname{Arg} z_2 < 0$).

12. If z is non-zero and $z = r(\cos\theta + i\sin\theta)$, then

$$z^{-1} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)).$$

13. **de Moivre's Theorem**

If n is an integer and θ is a real number, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Section 3: Solving Equations with Complex Numbers

1. Suppose that w is a non-zero complex number and $n \geq 2$. Each solution of $z^n = w$ is called an **nth root** of w.

2. Let

$$w = \rho(\cos\phi + i\sin\phi)$$

be a non-zero complex number in polar form. Then w has exactly n nth roots, given by

$$z_k = \rho^{1/n}\left(\cos\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right)\right), \quad k = 0, 1, \ldots, n-1.$$

These n nth roots form the vertices of an n-sided regular polygon inscribed in a circle of radius $\rho^{1/n}$ centred at 0.

3. If $w = \rho(\cos\phi + i\sin\phi)$, where ϕ is the principal argument of w, then

$$z_0 = \rho^{1/n}\left(\cos\frac{\phi}{n} + i\sin\frac{\phi}{n}\right)$$

is called the **principal nth root** of w, denoted by $\sqrt[n]{w}$ or $w^{1/n}$.

4. By an **nth root of unity** we mean an nth root of 1. These n roots are given by

$$z_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, \quad k = 0, 1, 2, \ldots, n-1.$$

5. **Strategy for finding nth roots**

To find the n nth roots $z_0, z_1, \ldots, z_{n-1}$ of a non-zero complex number w:

(a) express w in polar form, with modulus ρ and argument ϕ;

(b) substitute the values of ρ and ϕ in the formula

$$z_k = \rho^{1/n}\left(\cos\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) + \sin\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right)\right), \quad k = 0, 1, \ldots, n-1;$$

(c) if required, convert the roots to Cartesian form.

6. The two roots of $az^2 + bz + c = 0$, where a, b, c are in $\mathbb{C}$ and $a \neq 0$, are

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In certain cases a polynomial equation of degree n can be solved by substitution to obtain a quadratic equation.

Section 4: Sets of Complex Numbers

1. Inequalities between complex numbers have no meaning, unless the complex numbers are real. Since $|z|$, $\operatorname{Re} z$, $\operatorname{Im} z$ and $\operatorname{Arg} z$ are real, we can form inequalities involving them.

2. The following subsets of $\mathbb{C}$ are commonly used.

(a) Open half-plane $\{z : a\operatorname{Re} z + b\operatorname{Im} z > c\}$,
closed half-plane $\{z : a\operatorname{Re} z + b\operatorname{Im} z \geq c\}$,
where $a, b, c \in \mathbb{R}$ and a, b are not both zero.

(b) Open disc $\{z : |z - \alpha| < r\}$,
closed disc $\{z : |z - \alpha| \leq r\}$,
where $r > 0$.

(c) Open annulus $\{z : r_1 < |z - \alpha| < r_2\}$,
closed annulus $\{z : r_1 \leq |z - \alpha| \leq r_2\}$,
where $r_2 \geq r_1 \geq 0$.

(d) Ray (half-line) $\{z : \operatorname{Arg} z = \theta\}$,
where $\theta \in]-\pi, \pi]$.

(e) Open sector $\{z : a < \operatorname{Arg}(z - \alpha) < b\}$,
where $\alpha \in \mathbb{C}$ and $-\pi \leq a < b \leq \pi$.

A special case is the cut plane

$$\{z : |\operatorname{Arg} z| < \pi\} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$$

Section 5: Proving Inequalities

1. $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$.

2. **Triangle Inequality**

If $z_1, z_2 \in \mathbb{C}$, then

(a) $|z_1 + z_2| \leq |z_1| + |z_2|$ (usual form);

(b) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ (backwards form).

3. If $z, z_1, z_2, \ldots, z_n \in \mathbb{C}$, then

(a) $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$;

(b) $|z_1 - z_2| \leq |z_1| + |z_2|$;

(c) $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$;

(d) $|z_1 \pm z_2 \pm \cdots \pm z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$;

(e) $|z_1 \pm z_2 \pm \cdots \pm z_n| \geq |z_1| - |z_2| - \cdots - |z_n|$.

Unit A2 Complex Functions

Section 1: What is a Complex Function?

1. A **complex function** f is defined by specifying

(a) two sets A and B in the complex plane $\mathbb{C}$, and

(b) a rule which associates with each number z in A a unique number w in B; we write $w = f(z)$.

The set A is called the **domain** of the function f and B is called the **codomain** of f. The number w is called the **image of z under f**, or the **value of f at z**.

2. **Convention** When a function f is specified *just* by its rule, it is to be understood that the domain of f is the set of all complex numbers to which the rule is applicable, and the codomain of f is $\mathbb{C}$.

3. Given a function $f : A \longrightarrow B$, the **image** of f is

$$f(A) = \{f(z) : z \in A\}.$$

If $f(A) = B$, then f is called **onto**.

4. If $f(A) \subseteq \mathbb{R}$, then f is called **real-valued**.

If $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$, then f is a **real function**.

5. Let $f : A \longrightarrow \mathbb{C}$ and $g : B \longrightarrow \mathbb{C}$ be functions. Then the **sum** $f + g$ is the function with domain $A \cap B$ and rule

$$(f + g)(z) = f(z) + g(z);$$

the **multiple** λf, where $\lambda \in \mathbb{C}$, is the function with domain A and rule

$$(\lambda f)(z) = \lambda f(z);$$

the **product** fg is the function with domain $A \cap B$ and rule

$$(fg)(z) = f(z)g(z);$$

the **quotient** f/g is the function with domain $A \cap B - \{z : g(z) = 0\}$ and rule

$$(f/g)(z) = f(z)/g(z).$$

6. A **polynomial function** of degree n is defined by

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

where $a_0, \ldots, a_n \in \mathbb{C}$, and $a_n \neq 0$.

A **rational function** r is the quotient of two polynomial functions p and q: $r(z) = \dfrac{p(z)}{q(z)}$.

7. Let $f : A \longrightarrow \mathbb{C}$ and $g : B \longrightarrow \mathbb{C}$ be complex functions. Then the **composite function** $g \circ f$ has domain

$$\{z \in A : f(z) \in B\}$$

and rule

$$(g \circ f)(z) = g(f(z)).$$

8. The function $f : A \longrightarrow B$ is **one-one** if the images under f of distinct points in A are also distinct; that is,

$$\text{if } z_1, z_2 \in A \text{ and } z_1 \neq z_2, \quad \text{then } f(z_1) \neq f(z_2).$$

Equivalently, if $w \in f(A)$ then there is a *unique* $z \in A$ such that $f(z) = w$.

A function which is not one-one is **many-one**.

9. Let $f : A \longrightarrow B$ be a one-one function. Then the **inverse function**, f^{-1}, of f has domain $f(A)$ and rule

$$f^{-1}(w) = z, \quad \text{where } w = f(z).$$

10. **Strategy for proving that a function f has an inverse function**

EITHER prove that f is one-one directly by showing that

$$\text{if } z_1 \neq z_2, \text{ then } f(z_1) \neq f(z_2)$$

OR determine the image $f(A)$ and show that for each $w \in f(A)$ there is a unique $z \in A$ such that

$$f(z) = w.$$

11. Reducing the domain of a function (but leaving the rule unchanged) gives a **restriction** of the function.

Section 2: Special Types of Complex Function

1. The functions $\operatorname{Re} f : z \longmapsto \operatorname{Re}(f(z))$ and $\operatorname{Im} f : z \longmapsto \operatorname{Im}(f(z))$ are called the **real** and the **imaginary parts** of f. $\operatorname{Re} f$ and $\operatorname{Im} f$ are real-valued functions with the same domain as f.

2. A **path** is a subset Γ of $\mathbb{C}$ which is the image of an associated continuous function $\gamma : I \longrightarrow \mathbb{C}$, where I is a real interval.

The function γ is called a **parametrization**. If

$$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I)$$

where ϕ and ψ are real functions, then the equations

$$x = \phi(t), \quad y = \psi(t) \quad (t \in I),$$

are called **parametric equations**.

If I is the closed interval $[a, b]$, then $\gamma(a)$ and $\gamma(b)$ are called the **initial point** and **final point** of Γ, respectively.

It may be possible to obtain the equation of Γ in terms of x and y alone by eliminating t from the equations $x = \phi(t)$ and $y = \psi(t)$.

3. **Standard parametrizations**

Set	Standard parametrization
Line through points α and β.	$\gamma(t) = (1-t)\alpha + t\beta$ $(t \in \mathbb{R})$
Line segment between points α and β.	$\gamma(t) = (1-t)\alpha + t\beta$ $(t \in [0,1])$
Circle with centre α, radius r: $\lvert z-\alpha\rvert = r$.	$\gamma(t) = \alpha + r(\cos t + i\sin t)$ $(t \in [0, 2\pi])$
Arc of circle with centre α, radius r.	$\gamma(t) = \alpha + r(\cos t + i\sin t)$ $(t \in [t_1, t_2])$
Ellipse in standard form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.	$\gamma(t) = a\cos t + ib\sin t$ $(t \in [0, 2\pi])$
Parabola in standard form: $y^2 = 4ax$.	$\gamma(t) = at^2 + 2iat$ $(t \in \mathbb{R})$
Right half of hyperbola in standard form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.	$\gamma(t) = a\cosh t + ib\sinh t$ $(t \in \mathbb{R})$

4. Given a function $f : A \longrightarrow B$ and a subset S of A, the **image under f of S** is

$$f(S) = \{f(z) : z \in S\}.$$

5. If f is a continuous function and Γ is a path in the domain of f, with parametrization γ, then $f(\Gamma)$ is called the **image path**; $f(\Gamma)$ has parametrization $f \circ \gamma$.

6. **Strategy for determining an image path**

Let the parametrization of the path Γ be

$$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I).$$

Then the image path $f(\Gamma)$ is found

EITHER

by using the geometric properties of the continuous function f

OR

by substituting $x = \phi(t)$, $y = \psi(t)$ into the equation

$$u + iv = f(x + iy)$$

and then, by equating real parts and imaginary parts, obtaining expressions for u and v in terms of t. (These expressions are the parametric equations of the image path $f(\Gamma)$, associated with the parametrization $f \circ \gamma$.)

Section 3: Images of Grids

A **Cartesian grid** consists of lines of the form $x = a$ and $y = b$, usually evenly spaced in both directions.

A **polar grid** consists of circles centred at 0 and rays emerging from 0. Each of the circles has an equation of the form $r = a$, where a is a positive constant, and each of the rays has an equation of the form $\theta = b$, where b is a constant in the interval $]-\pi, \pi]$.

Using the Strategy in Section 2 for determining an image path $f(\Gamma)$, we can obtain the images of Cartesian and polar grids.

Section 4: Exponential, Trigonometric and Hyperbolic Functions

1. For all $z = x + iy$ in $\mathbb{C}$,

$$e^z = e^x(\cos y + i\sin y).$$

The function

$$z \longmapsto e^z \quad (z \in \mathbb{C})$$

is called the **exponential function** and is denoted by exp.

If z is real, $z = x + 0i$, then

$$e^z = e^x(\cos 0 + i\sin 0) = e^x.$$

If z is imaginary, $z = 0 + iy$, then

$$e^{iy} = \cos y + i\sin y.$$

2. **Exponential identities**

(a) $e^{z_1+z_2} = e^{z_1}e^{z_2}$

(b) $|e^z| = e^{\operatorname{Re} z}$

(c) $e^{-z} = 1/e^z$

(d) $e^{z+2\pi i} = e^z$

3. **The geometric nature of exp**

(a) For all $n \in \mathbb{Z}$, $e^{z+2n\pi i} = e^z$, so each of the points $z + 2n\pi i$, $n \in \mathbb{Z}$, has the same image.

(b) The line $x = a$ is mapped to the path with parametric equations

$$u = e^a\cos t, \quad v = e^a\sin t \quad (t \in \mathbb{R}).$$

This is the circle with radius e^a and centre 0.

(c) The line $y = b$ is mapped to the path with parametric equations

$$u = e^t\cos b, \quad v = e^t\sin b \quad (t \in \mathbb{R}).$$

This is the ray from 0 (excluded) through $\cos b + i\sin b$.

(d) The strip $\{x + iy : -\pi < y \leq \pi\}$ is mapped to $\mathbb{C} - \{0\}$.

4. **Trigonometric functions**

The functions cos, sin, tan, sec, cot and cosec are defined as follows:

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right),$$
$$\tan z = \frac{\sin z}{\cos z}, \quad \sec z = \frac{1}{\cos z},$$
$$\cot z = \frac{\cos z}{\sin z}, \quad \operatorname{cosec} z = \frac{1}{\sin z}.$$

Both cos and sin have domain $\mathbb{C}$.

Since $\{z : \cos z = 0\} = \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$, both tan and sec have domain $\mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$.

Since $\{z : \sin z = 0\} = \{n\pi : n \in \mathbb{Z}\}$, both cot and cosec have domain $\mathbb{C} - \{n\pi : n \in \mathbb{Z}\}$.

5. All the identities satisfied by the real trigonometric functions also hold for the complex trigonometric functions.

6. **Hyperbolic functions**

The functions cosh, sinh, tanh, sech, coth and cosech are defined as follows:

$$\cosh z = \frac{1}{2}(e^z + e^{-z}), \quad \sinh z = \frac{1}{2}\left(e^z - e^{-z}\right),$$
$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z},$$
$$\coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{cosech} z = \frac{1}{\sinh z}.$$

Both cosh and sinh have domain $\mathbb{C}$.

Since $\{z : \cosh z = 0\} = \{(n + \frac{1}{2})\pi i : n \in \mathbb{Z}\}$, both tanh and sech have domain $\mathbb{C} - \{(n + \frac{1}{2})\pi i : n \in \mathbb{Z}\}$.

Since $\{z : \sinh z = 0\} = \{n\pi i : n \in \mathbb{Z}\}$, both coth and cosech have domain $\mathbb{C} - \{n\pi i : n \in \mathbb{Z}\}$.

7. For all z in $\mathbb{C}$,

$$\sin(iz) = i \sinh z \quad \text{and} \quad \cos(iz) = \cosh z.$$

8. All the identities satisfied by the real hyperbolic functions also hold for the complex hyperbolic functions.

Section 5: Logarithms and Powers

1. For $z \in \mathbb{C} - \{0\}$, the **principal logarithm** of z is

$$\operatorname{Log} z = \log_e |z| + i \operatorname{Arg} z.$$

The corresponding **principal logarithm function** is called Log. It is the inverse of

$$f(z) = e^z \quad (z \in \{x + iy : -\pi < y \leq \pi\}),$$

and satisfies

$$e^{\operatorname{Log} z} = z, \quad \text{for } z \in \mathbb{C} - \{0\},$$

and

$$\operatorname{Log}(e^z) = z, \quad \text{for } z \in \{x + iy : -\pi < y \leq \pi\}.$$

2. **Logarithmic identities**

(a) $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$, if $\operatorname{Arg} z_1, \operatorname{Arg} z_2 \in]-\frac{1}{2}\pi, \frac{1}{2}\pi]$;

(b) $\operatorname{Log}(1/z) = -\operatorname{Log} z$, if $\operatorname{Arg} z \in]-\pi, \pi[$.

Part (a) holds in the following form for any values in the domain, $\mathbb{C} - \{0\}$, of Log:

$$\operatorname{Log} z_1 z_2 = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2n\pi i,$$

where n is -1, 0 or 1 according as $\operatorname{Arg} z_1 + \operatorname{Arg} z_2$ is greater than π, lies in the interval $]-\pi, \pi]$, or is less than or equal to $-\pi$.

(c) For $z, \alpha \in \mathbb{C}$, with $z \neq 0$, the **principal αth power** of z is

$$z^\alpha = \exp(\alpha \operatorname{Log} z).$$

This definition agrees with the usual meaning of z^α, for $\alpha = n$ or $\alpha = 1/n$, where $n \in \mathbb{N}$.

The function $z \longmapsto z^\alpha$ is called the **principal αth power function.**

Unit A3 Continuity

Section 1: Sequences

1. A **(complex) sequence** is an unending list of complex numbers

$$z_1, z_2, z_3, \ldots .$$

The complex number z_n is called the **nth term of the sequence**, and the sequence is denoted by $\{z_n\}$.

2. The sequence $\{z_n\}$ is **convergent with limit α**, or **converges to α**, or **tends to α**, if for each positive number ε, there is an integer N such that

$$|z_n - \alpha| < \varepsilon, \quad \text{for all } n > N.$$

If $\{z_n\}$ converges to α, then we write

EITHER $\lim_{n\to\infty} z_n = \alpha$

OR $z_n \to \alpha$ as $n \to \infty$.

If the limit α is 0, then $\{z_n\}$ is called a **null sequence**.

3. (a) The sequence $\{z_n\}$ converges to α if and only if the sequence $\{z_n - \alpha\}$ is null.

(b) If a sequence $\{z_n\}$ is convergent, then it has a *unique* limit.

(c) A **constant sequence**

$$z_n = \alpha, \quad n = 1, 2, \ldots,$$

is convergent with limit α.

(d) If a given sequence converges to α, then this remains true if we *add, delete, or alter a finite number of terms.*

4. Squeeze Rule

If $\{a_n\}$ is a real null sequence of non-negative terms, and if

$$|z_n| \le a_n, \quad \text{for } n = 1, 2, \ldots,$$

then $\{z_n\}$ is a null sequence.

5. Basic null sequences

The following sequences are null:

(a) $\{1/n^p\}$, for $p > 0$;

(b) $\{\alpha^n\}$, for $|\alpha| < 1$.

6. Combination Rules

If $\lim_{n\to\infty} z_n = \alpha$ and $\lim_{n\to\infty} w_n = \beta$, then

Sum Rule $\lim_{n\to\infty} (z_n + w_n) = \alpha + \beta$;

Multiple Rule $\lim_{n\to\infty} (\lambda z_n) = \lambda\alpha$, where $\lambda \in \mathbb{C}$;

Product Rule $\lim_{n\to\infty} (z_n w_n) = \alpha\beta$;

Quotient Rule $\lim_{n\to\infty} \left(\frac{z_n}{w_n}\right) = \frac{\alpha}{\beta}$, provided that $\beta \neq 0$.

7. If $\lim_{n\to\infty} z_n = \alpha$, then

(a) $\lim_{n\to\infty} |z_n| = |\alpha|$;

(b) $\lim_{n\to\infty} \overline{z_n} = \overline{\alpha}$;

(c) $\lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} \alpha$;

(d) $\lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} \alpha$.

8. A sequence which is not convergent is **divergent.**

9. The sequence $\{z_n\}$ **tends to infinity** if, for each positive number M, there is an integer N such that

$$|z_n| > M, \quad \text{for all } n > N.$$

In this case we write

$$z_n \to \infty \text{ as } n \to \infty.$$

10. **Reciprocal Rule**

Let $\{z_n\}$ be a sequence. Then

$$z_n \to \infty \text{ as } n \to \infty$$

if and only if

$\{1/z_n\}$ is a null sequence.

11. Let $\{n_k\}$ be a sequence of positive integers which are strictly increasing; that is,

$$n_1 < n_2 < n_3 < \ldots .$$

Then the sequence $\{z_{n_k}\}$ is a **subsequence** of the sequence $\{z_n\}$.

In particular, $\{z_{2k}\}$ is the **even subsequence** and $\{z_{2k-1}\}$ is the **odd subsequence.**

12. **Subsequence Rules**

(a) **First Subsequence Rule** The sequence $\{z_n\}$ is divergent if $\{z_n\}$ has two convergent subsequences with different limits.

(b) **Second Subsequence Rule** The sequence $\{z_n\}$ is divergent if $\{z_n\}$ has a subsequence which tends to infinity.

13. (a) If $|\alpha| > 1$, then the sequence $\{\alpha^n\}$ tends to infinity.

(b) If $|\alpha| = 1$ and $\alpha \neq 1$, then the sequence $\{\alpha_n\}$ is divergent.

14. If $\{z_n\}$ is a convergent sequence, then there is a positive number M such that

$$|z_n| \le M, \quad \text{for } n = 1, 2, \ldots .$$

In this case, we say that $\{z_n\}$ is **bounded.**

Section 2: Continuous Functions

1. Continuity: sequential definition

Let $f : A \longrightarrow \mathbb{C}$ and $\alpha \in A$. Then f is **continuous at α** if, for each sequence $\{z_n\}$ in A such that $z_n \to \alpha$, we have

$$f(z_n) \to f(\alpha);$$

that is,

$$z_n \to \alpha \implies f(z_n) \to f(\alpha).$$

If f is continuous at each α in A, then we say that f is **continuous** (on A).

If f is not continuous at α, then we say that f is **discontinuous** at α.

2. Continuity: ε-δ definition

Let $f : A \longrightarrow \mathbb{C}$ and $\alpha \in A$. Then f is **continuous at α** if, for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$z \in A,\ |z - \alpha| < \delta \implies |f(z) - f(\alpha)| < \varepsilon.$$

3. The ε-δ definition of continuity is equivalent to the sequential definition of continuity.

4. Basic continuous functions

(a) polynomial and rational functions;

(b) $f(z) = |z|$, $\overline{z}$, $\operatorname{Re} z$, $\operatorname{Im} z$;

(c) $f(z) = e^z$;

(d) trigonometric and hyperbolic functions;

(e) $f(z) = \operatorname{Arg} z$, $\operatorname{Log} z$, z^α, on $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$.

5. Combination Rules

If the functions f and g are continuous at α, then so are

Sum Rule	$f + g$;
Multiple Rule	λf, for $\lambda \in \mathbb{C}$;
Product Rule	fg;
Quotient Rule	f/g, provided that $g(\alpha) \neq 0$.

6. Composition Rule

If the function f is continuous at α, and if the function g is continuous at $f(\alpha)$, then $g \circ f$ is continuous at α.

7. Restriction Rule

If the function f has domain A, the function g has domain B and

1. f is the restriction of g to A;
2. g is continuous at $\alpha \in A$,

then f is continuous at α.

Section 3: Limits of Functions

1. The point α is a **limit point** of a set A if there is a sequence $\{z_n\}$ such that

$$z_n \in A - \{\alpha\}, \quad \text{for } n = 1, 2, \ldots,$$

and

$$\lim_{n\to\infty} z_n = \alpha.$$

2. Let f be a function with domain A, and suppose that α is a limit point of A. Then the function f has **limit β as z tends to α** if for each sequence $\{z_n\}$ in $A - \{\alpha\}$ such that $z_n \to \alpha$,

$$f(z_n) \to \beta.$$

In this case we write

EITHER $\lim_{z\to\alpha} f(z) = \beta$,

OR $f(z) \to \beta$ as $z \to \alpha$.

3. Strategy for proving that a limit does not exist

To prove that $\lim_{z\to\alpha} f(z)$ does not exist, where α is a limit point of the domain A of the function f:

EITHER

(a) find two sequences $\{z_n\}$ and $\{z_n'\}$ in $A - \{\alpha\}$ which both tend to α, such that the sequences $\{f(z_n)\}$ and $\{f(z_n')\}$ have different limits;

OR

(b) find a sequence $\{z_n\}$ in $A - \{\alpha\}$ which tends to α, such that the sequence $\{f(z_n)\}$ tends to infinity.

4. Let f be a function with domain A and suppose that $\alpha \in A$ is a limit point of A. Then

$$f \text{ is continuous at } \alpha \iff \lim_{z\to\alpha} f(z) = f(\alpha).$$

5. Let f be a function with domain A and suppose that α is a limit point of A. Then the function f has **limit β as z tends to α** if for each positive ε, there is a positive δ such that

$$|f(z) - \beta| < \varepsilon, \quad \text{for all } z \in A - \{\alpha\} \text{ with } |z - \alpha| < \delta.$$

6. Combination Rules

Let f and g be functions with domains A and B, respectively, and suppose that α is a limit point of $A \cap B$. If

$$\lim_{z\to\alpha} f(z) = \beta \quad \text{and} \quad \lim_{z\to\alpha} g(z) = \gamma,$$

then

Sum Rule	$\lim_{z\to\alpha} (f(z) + g(z)) = \beta + \gamma$;
Multiple Rule	$\lim_{z\to\alpha} (\lambda f(z)) = \lambda\beta$, for $\lambda \in \mathbb{C}$;
Product Rule	$\lim_{z\to\alpha} (f(z)g(z)) = \beta\gamma$;
Quotient Rule	$\lim_{z\to\alpha} (f(z)/g(z)) = \beta/\gamma$, provided that $\gamma \neq 0$.

Section 4: Regions

1. A set A in $\mathbb{C}$ is **open** if each point α in A is the centre of some open disc lying entirely in A.

2. **Combination Rules**

If A_1 and A_2 are open sets, then so are:

(a) $A_1 \cup A_2$ and

(b) $A_1 \cap A_2$.

This rule extends to n open sets $A_1, A_2, \ldots, A_n$:

$A_1 \cup A_2 \cup \ldots \cup A_n$ and $A_1 \cap A_2 \cap \ldots \cap A_n$

are open sets.

3. A set A in $\mathbb{C}$ is **(pathwise) connected** if each pair of points α, β in A can be joined by a path lying entirely in A.

4. Connected sets in which any two points α and β can be joined by a line segment are called **convex**.

5. Let f be a continuous function whose domain A is connected. Then the image $f(A)$ is also connected.

6. A **region** is a non-empty, connected, open set.

7. **Basic regions**

(a) any open disc;

(b) any open half-plane;

(c) the complement of any closed disc;

(d) any open annulus;

(e) any open rectangle;

(f) any open sector (including cut planes);

(g) the set $\mathbb{C}$ itself.

8. If $\mathcal{R}$ is a region and $\alpha_0 \in \mathcal{R}$, then $\mathcal{R} - \{\alpha_0\}$ is also a region.

Section 5: The Extreme Value Theorem

1. A set E in $\mathbb{C}$ is **closed** if its complement $\mathbb{C} - E$ is open.

2. If E is a closed set and $\{z_n\}$ is a sequence in E with limit α, then $\alpha \in E$.

3. **Combination Rules**

If E_1 and E_2 are closed sets, then so are:

(a) $E_1 \cup E_2$ and

(b) $E_1 \cap E_2$.

This rule extends to n closed sets $E_1, E_2, \ldots, E_n$:

$E_1 \cup E_2 \cup \ldots \cup E_n$ and $E_1 \cap E_2 \cap \ldots \cap E_n$

are closed sets.

4. A set E in $\mathbb{C}$ is **bounded** if it is contained in some closed disc.

A set is **unbounded** if it is not bounded.

5. A set E in $\mathbb{C}$ is **compact** if E is closed and bounded.

6. **Extreme Value Theorem**

Let the function f be continuous on a compact set E. Then there are numbers α, β in E such that

$$|f(\beta)| \leq |f(z)| \leq |f(\alpha)|, \quad \text{for all } z \in E.$$

7. **Boundedness Theorem**

Let the function f be continuous on a compact set E. Then there is a number M such that

$$|f(z)| \leq M, \quad \text{for all } z \in E.$$

8. If the function f is continuous on a compact set E, then $f(E)$ is compact.

9. Let A be a subset of $\mathbb{C}$ and let $\alpha \in \mathbb{C}$. Then

(a) α is an **interior point** of A if there is an open disc centred at α which lies entirely in A;

(b) α is an **exterior point** of A if there is an open disc centred at α which lies entirely outside A.

The set of interior points of A forms the **interior** of A, written $\operatorname{int} A$, and the set of exterior points of A forms the **exterior** of A, written $\operatorname{ext} A$.

10. Let A be a subset of $\mathbb{C}$ and let $\alpha \in \mathbb{C}$. Then α is a **boundary point** of A if each open disc centred at α contains at least one point of A and at least one point of $\mathbb{C} - A$.

The set of boundary points of A forms the **boundary** of A, written ∂A.

The sets $\operatorname{int} A$, $\operatorname{ext} A$ and ∂A are disjoint and

$$\partial A = \mathbb{C} - (\operatorname{int} A \cup \operatorname{ext} A).$$

Also $\operatorname{int} A$ and $\operatorname{ext} A$ are open, whereas ∂A is closed.

11. **Nested Rectangles Theorem**

Let $R_n, n = 0, 1, 2, \ldots$, be a sequence of closed rectangles with sides parallel to the axes, and with diagonals of lengths $s_n, n = 0, 1, 2, \ldots$, such that

1. $R_0 \supseteq R_1 \supseteq R_2 \supseteq \ldots$, and
2. $\lim_{n \to \infty} s_n = 0$.

Then there is a unique complex number α which lies in all of the rectangles R_n. Moreover, for each positive number ε, there is an integer N such that

$$R_n \subseteq \{z : |z - \alpha| < \varepsilon\}, \quad \text{for all } n > N.$$

Unit A4 Differentiation

Section 1: Derivatives of Complex Functions

1. Let f be a complex function whose domain contains the point α. Then the **derivative of f at α** is

$$\lim_{z\to\alpha}\frac{f(z)-f(\alpha)}{z-\alpha}\quad\left(\text{or}\quad\lim_{h\to 0}\frac{f(\alpha+h)-f(\alpha)}{h}\right),$$

provided that this limit exists. If it does exist, then f **is differentiable at α**. If f is differentiable at every point of a set A, then f **is differentiable on A**. A function **is differentiable** if it is differentiable on its domain.

The derivative of f at α is denoted by $f'(\alpha)$, and the function

$$f' : z \longmapsto f'(z)$$

is called the **derivative** of f. The domain of f' is the set of all complex numbers at which f is differentiable.

2. A function is **entire** if it is differentiable on the whole of $\mathbb{C}$.

3. If a function f is differentiable on a region $\mathcal{R}$, then f is said to be **analytic on $\mathcal{R}$**. If the domain of f is a region and if f is differentiable on its domain, then f is said to be **analytic**. A function f is **analytic at a point α** if it is differentiable on a region containing α.

4. If the complex function f is differentiable at α, then f is continuous at α.

5. **Linear Approximation Theorem**

If the complex function f is differentiable at α, then f may be approximated near α by a linear polynomial. More precisely,

$$f(z) = f(\alpha) + (z-\alpha)f'(\alpha) + e(z),$$

where e is an 'error' function satisfying $e(z)/(z-\alpha) \to 0$ as $z \to \alpha$.

6. **Combination Rules**

Let f and g be complex functions with domains A and B, respectively, and let α be a limit point of $A \cap B$. If f and g are differentiable at α, then

Sum Rule $(f+g)'(\alpha) = f'(\alpha) + g'(\alpha)$;

Multiple Rule $(\lambda f)'(\alpha) = \lambda f'(\alpha)$;

Product Rule $(fg)'(\alpha) = f'(\alpha)g(\alpha) + f(\alpha)g'(\alpha)$;

Quotient Rule $(f/g)'(\alpha) = \dfrac{g(\alpha)f'(\alpha) - f(\alpha)g'(\alpha)}{(g(\alpha))^2}$,

provided that $g(\alpha) \neq 0$.

7. A polynomial function

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n \quad (z \in \mathbb{C}),$$

where $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$, is entire with derivative

$$p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1} \quad (z \in \mathbb{C}).$$

Any rational function is analytic.

8. **Strategy A for non-differentiability**

If f is discontinuous at α, then f is not differentiable at α.

9. **Strategy B for non-differentiability**

To prove that a function f is not differentiable at α, apply the strategy for the non-existence of limits to the difference quotient

$$\frac{f(z)-f(\alpha)}{z-\alpha}.$$

10. **Higher-order derivatives** of a function f are obtained by repeated differentiation:

$$(f')' = f'' = f^{(2)},$$
$$(f'')' = f''' = f^{(3)},$$

and so on.

The **nth derivative** of f is denoted by $f^{(n)}$.

11. **A geometric interpretation of derivatives**

If $f'(\alpha) \neq 0$, then, to a close approximation,

$$f(z) - f(\alpha) \cong f'(\alpha)(z-\alpha).$$

It follows that, to a close approximation, a small disc centred at α is mapped to a small disc centred at $f(\alpha)$. In this process, the disc is rotated through the angle $\operatorname{Arg} f'(\alpha)$, and scaled by the factor $|f'(\alpha)|$.

Section 2: The Cauchy–Riemann Equations

1. Cauchy–Riemann Theorem

If f is differentiable at $\alpha = a + ib$, then $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ exist at (a, b) and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x}(a,b) = \frac{\partial v}{\partial y}(a,b) \quad \text{and} \quad \frac{\partial v}{\partial x}(a,b) = -\frac{\partial u}{\partial y}(a,b).$$

2. Strategy C for non-differentiability

Let $f(x + iy) = u(x, y) + iv(x, y)$. If either

$$\frac{\partial u}{\partial x}(a,b) \neq \frac{\partial v}{\partial y}(a,b) \quad \text{or} \quad \frac{\partial v}{\partial x}(a,b) \neq -\frac{\partial u}{\partial y}(a,b),$$

then f is not differentiable at $a + ib$.

3. Cauchy–Riemann Converse Theorem

Let $f(x + iy) = u(x, y) + iv(x, y)$ be defined on a region $\mathcal{R}$ containing $a + ib$. If the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$

1. exist on $\mathcal{R}$,
2. are continuous at (a, b),
3. satisfy the Cauchy–Riemann equations at (a, b),

then f is differentiable at $a + ib$ and

$$f'(a+ib) = \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b).$$

Section 3: The Composition, Inverse and Restriction Rules

1. Composition Rule

Let f and g be complex functions, and let α be a limit point of the domain of $g \circ f$. If f is differentiable at α, and g is differentiable at $f(\alpha)$, then $g \circ f$ is differentiable at α, and

$$(g \circ f)'(\alpha) = g'(f(\alpha))f'(\alpha).$$

2. Inverse Function Rule

Let $f : A \longrightarrow B$ be a one-one complex function, and suppose that f^{-1} is continuous at $\beta \in B$. If f has a non-zero derivative at $f^{-1}(\beta) \in A$, then f^{-1} is differentiable at β and $\left(f^{-1}\right)'(\beta) = \dfrac{1}{f'\left(f^{-1}(\beta)\right)}$.

3. Restriction Rule

Let f and g be functions with domains A and B, respectively, and let $A \subseteq B$. If $\alpha \in A$ is a limit point of A and

1. $f(z) = g(z)$, for $z \in A$,
2. g is differentiable at α,

then f is differentiable at α, and $f'(\alpha) = g'(\alpha)$.

4. Standard derivatives

Rule of f	Rule of f'	Domain of f'
$\alpha,\ \alpha \in \mathbb{C}$	0	$\mathbb{C}$
$z^k,\ k \in \mathbb{Z},\ k > 0$	kz^{k-1}	$\mathbb{C}$
$z^k,\ k \in \mathbb{Z},\ k < 0$	kz^{k-1}	$\mathbb{C} - \{0\}$
$z^\alpha,\ \alpha \in \mathbb{C} - \mathbb{Z}$	$\alpha z^{\alpha-1}$	$\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$
$\exp z$	$\exp z$	$\mathbb{C}$
$\operatorname{Log} z$	$1/z$	$\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$
$\sin z$	$\cos z$	$\mathbb{C}$
$\cos z$	$-\sin z$	$\mathbb{C}$
$\tan z$	$\sec^2 z$	$\mathbb{C} - \left\{\left(n + \frac{1}{2}\right)\pi : n \in \mathbb{Z}\right\}$
$\sinh z$	$\cosh z$	$\mathbb{C}$
$\cosh z$	$\sinh z$	$\mathbb{C}$
$\tanh z$	$\operatorname{sech}^2 z$	$\mathbb{C} - \left\{\left(n + \frac{1}{2}\right)\pi i : n \in \mathbb{Z}\right\}$

Section 4: Smooth Paths

1. Let Γ be a path with parametrization $\gamma : I \longrightarrow \mathbb{C}$, and suppose that $c \in I$. If γ is differentiable at c and if $\gamma'(c) \neq 0$, then $\gamma'(c)$ may be interpreted as a tangent vector to the path Γ at the point $\gamma(c)$.

2. Let ϕ and ψ be real functions, both with domain some interval I. Then the parametrization

$$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I)$$

is differentiable at a point $c \in I$ if and only if both ϕ and ψ are differentiable at c. If ϕ and ψ are differentiable at c, then

$$\gamma'(c) = \phi'(c) + i\psi'(c).$$

3. A parametrization $\gamma : I \longrightarrow \mathbb{C}$ is **smooth** if

(a) γ is differentiable on I;

(b) γ' is continuous on I;

(c) γ' is non-zero on I.

A path is **smooth** if its parametrization is smooth.

4. Let f be a function that is analytic on a region $\mathcal{R}$, and suppose that $f'(\alpha) \neq 0$ for some $\alpha \in \mathcal{R}$. If $\Gamma : \gamma$ is a smooth path in $\mathcal{R}$, passing through α, then the tangent vector to $f(\Gamma)$ at $f(\alpha)$ may be obtained from the tangent vector to Γ at α by a rotation through the angle $\operatorname{Arg} f'(\alpha)$ and a scaling by the factor $|f'(\alpha)|$.

5. A function is **conformal at α** if it leaves unchanged the size and orientation of the angle between any two smooth paths through α. A function is **conformal on a set S** if it is conformal at every point of S. A function is **conformal** if it is conformal on its domain.

6. Let f be a function which is analytic at α, with $f'(\alpha) \neq 0$. Then f is conformal at α.

7. Smooth paths that meet at right angles are said to be **orthogonal**. A grid made up of orthogonal smooth paths is called an **orthogonal grid**.

PART III: INDEX

absolute value 10
analytic function 18
annulus 11
argument 10
  principal 10

basic continuous functions 16
basic null sequences 15
Binomial Theorem 9
boundary of a set 17
boundary point 17
bounded set 17
Boundedness Theorem 17

Cauchy–Riemann Converse Theorem 19
Cauchy–Riemann equations 19
Cauchy–Riemann Theorem 19
closed set 17
Combination Rules
  for closed sets 17
  for continuity 16
  for differentiability 18
  for limits 16
  for open sets 17
  for sequences 15
compact set 17
complex conjugate 9
  properties of 9
complex numbers 9
  Cartesian form 9
  equality 9
  imaginary part 9
  negative of 9
  polar form 10
  quotient of 9
  real part 9
  reciprocal of 9
complex plane 10
Composition Rule
  for continuity 16
  for differentiability 19
conformal function 19
connected set 17
continuous function
  ε-δ definition 16
  sequential definition 16
convergent sequence 15
convex set 17
cut plane 11

de Moivre's Theorem 10
derivatives 18
  geometric interpretation 18
  higher order 18
  standard 19
differentiable function 18
discontinuous function 16
discs 11
divergent sequence 15

entire function 18
exponential function 13
  geometric nature 13
exponential identities 13
exterior point of a set 17
Extreme Value Theorem 17

functions (complex) 12
  codomain of 12
  composition of 12
  domain of 12
  real-valued 12
function (real) 12

Geometric Series Identity 9
grid
  Cartesian 13
  orthogonal 19
  polar 13

half-line 11
half-plane 11
hyperbolic functions 14
hyperbolic identities 14

image of a function 12
image of a point 12
image of a set 13
image path 13
imaginary part function 12
imaginary part of a complex number 9
interior point of a set 17
inverse function 12
Inverse Function Rule 19

limit of a function as $z \to \alpha$ 16
limit of a sequence 15
limit point 16
Linear Approximation Theorem 18
logarithm function, principal 14
logarithmic identities 14

many-one function 12
modulus 10
  properties of 10

Nested Rectangles Theorem 17
null sequence 15

one-one function 12
onto function 12
open set 17

parametric equations 12
parametrization 12
  smooth 19
  standard 13
path 12
  final point 12
  initial point 12
  smooth 19
pathwise connected set 17
polar coordinates 10
polynomial function 12
principal logarithm function 14
principal nth root 11
principal power function 14

rational function 12
ray 11
real part function 12
real part of a complex number 9
Reciprocal Rule 15
region 17
  basic 17
restriction of a function 12
Restriction Rule
  for continuity 16
  for differentiability 19
roots of a complex number 11
roots of unity 11

sector 11
sequence 15
  bounded 15
  constant 15
  convergent 15
  divergent 15
  null 15
  tending to infinity 15
smooth parametrization 19
smooth path 19
Squeeze Rule for sequences 15
subsequence 15
  even 15
  odd 15
Subsequence Rules 15

term of sequence 15
Triangle Inequality 11
trigonometric functions 14
trigonometric identities 14

unbounded set 17

value of a function 12

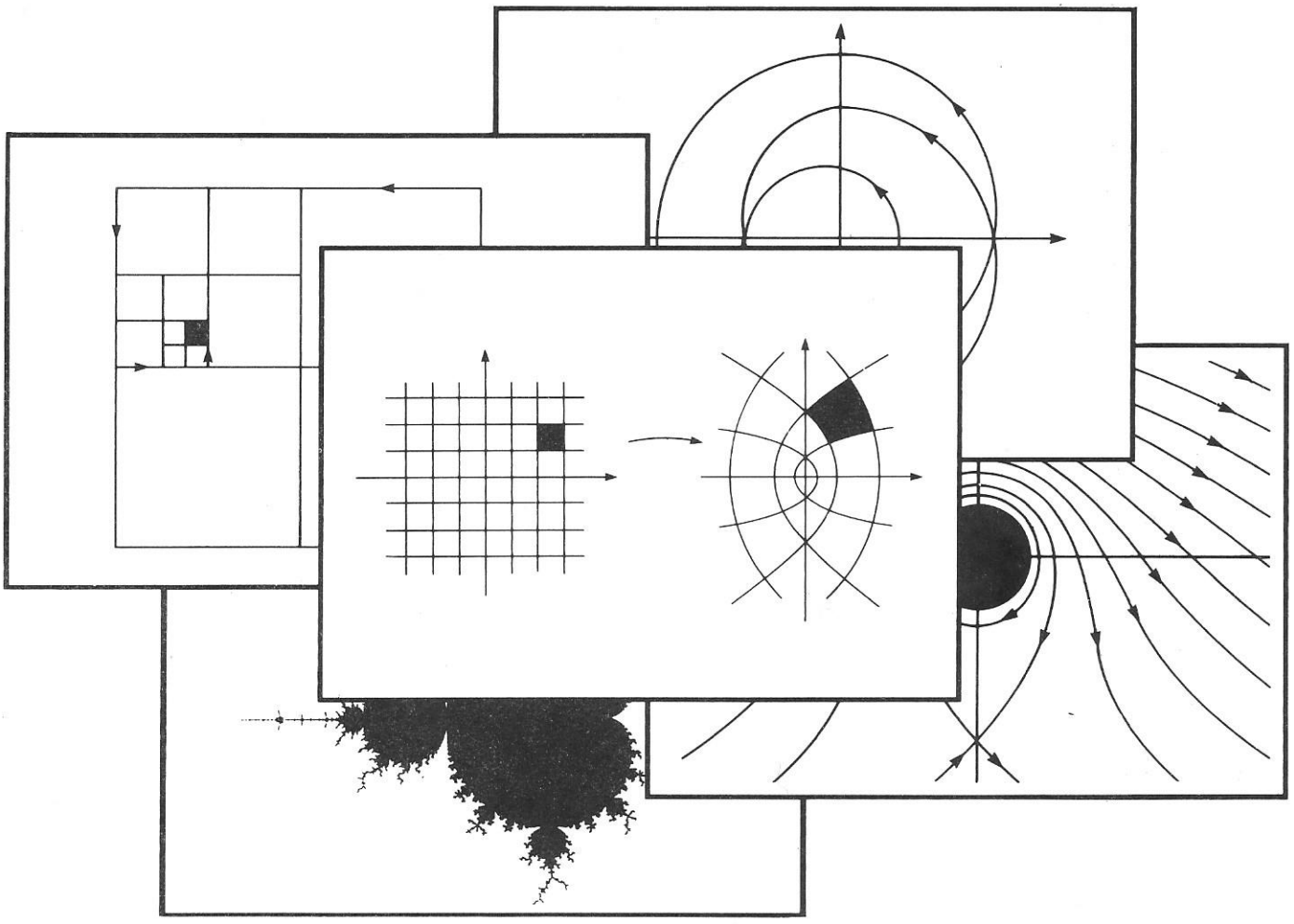
Printed in the United Kingdom by
The Open University

M337 HB2
Mathematics: A Third Level Course

The Open University

COMPLEX ANALYSIS

HANDBOOK TWO



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CONTENTS

Part I: Unit Summaries 3

Unit B1 Integration 3

Unit B2 Cauchy's Theorem 5

Unit B3 Taylor Series 6

Unit B4 Laurent Series 10

Part II: Index 12

PART I: UNIT SUMMARIES

Unit B1 Integration

Section 1: Integrating Real Functions

This section revises the definition and main properties of the integration of real functions.

Section 2: Integrating Complex Functions

1. Let $\Gamma : \gamma(t)$ $(t \in [a,b])$ be a smooth path in $\mathbb{C}$, and let f be a function which is continuous on Γ. Then the **integral of f along the path Γ**, denoted by $\int_\Gamma f(z)\,dz$ or $\int_\Gamma f$, is

$$\int_\Gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

This integral is evaluated as follows:

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt,$$

where $u(t) = \mathrm{Re}(f(\gamma(t))\gamma'(t))$ and $v(t) = \mathrm{Im}\,(f(\gamma(t))\gamma'(t))$.

2. The above definition may be remembered by writing

$$z = \gamma(t), \quad dz = \gamma'(t)\,dt.$$

3. Two parametrizations

$$\gamma_1 : [a_1,b_1] \longrightarrow \mathbb{C} \quad \text{and} \quad \gamma_2 : [a_2,b_2] \longrightarrow \mathbb{C}$$

are **equivalent** if there is a function $h : [a_1,b_1] \longrightarrow [a_2,b_2]$ satisfying

(a) $h(a_1) = a_2, h(b_1) = b_2$,

(b) h' exists on $[a_1,b_1]$, and is continuous and positive there,

such that

$$\gamma_1(t) = \gamma_2(h(t)), \quad \text{for } t \in [a_1,b_1].$$

If γ_1 and γ_2 are equivalent parametrizations, then the paths $\Gamma : \gamma_1$ and $\Gamma : \gamma_2$ are **equivalent paths**.

4. If the smooth paths $\Gamma : \gamma_1$ and $\Gamma : \gamma_2$ are equivalent and the function f is continuous on Γ, then $\int_\Gamma f(z)\,dz$ does not depend on which parametrization, γ_1 or γ_2, is used.

5. A **contour** Γ is a path which can be subdivided into a finite number of smooth paths $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, joined end to end; the order of these constituent smooth paths is indicated by writing

$$\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n.$$

The **initial point** of Γ is the initial point of Γ_1, and the **final point** of Γ is the final point of Γ_n.

6. Let $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ be a contour, and let f be a function which is continuous on Γ. Then the **(contour) integral of f along Γ**, denoted by $\int_\Gamma f(z)\,dz$ or $\int_\Gamma f$, is

$$\int_\Gamma f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \cdots + \int_{\Gamma_n} f(z)\,dz.$$

7. All splittings of a contour into smooth paths give the same value for the contour integral.

8. **Combination Rules**
Let Γ be a contour, and let f and g be functions which are continuous on Γ. Then

Sum Rule

$$\int_\Gamma (f(z)+g(z))\,dz = \int_\Gamma f(z)\,dz + \int_\Gamma g(z)\,dz;$$

Multiple Rule

$$\int_\Gamma \lambda f(z)\,dz = \lambda \int_\Gamma f(z)\,dz, \quad \text{for } \lambda \in \mathbb{C}.$$

9. Let $\Gamma : \gamma(t)$ $(t \in [a,b])$ be a smooth path. Then the **reverse path**, denoted by $\widetilde{\Gamma}$, is the path with parametrization $\widetilde{\gamma}(t)$, where

$$\widetilde{\gamma}(t) = \gamma(a+b-t) \quad (t \in [a,b]).$$

As sets, Γ and $\widetilde{\Gamma}$ are the same.

10. If $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ is a contour, then the **reverse contour** $\widetilde{\Gamma}$ of Γ is

$$\widetilde{\Gamma} = \widetilde{\Gamma}_n + \widetilde{\Gamma}_{n-1} + \cdots + \widetilde{\Gamma}_1.$$

11. **Reverse Contour Theorem**
Let Γ be a contour, and let f be a function which is continuous on Γ. Then, if $\widetilde{\Gamma}$ is the reverse contour of Γ,

$$\int_{\widetilde{\Gamma}} f(z)\,dz = -\int_\Gamma f(z)\,dz.$$

Section 3: Evaluating Contour Integrals

1. Let f and F be functions defined on a region $\mathcal{R}$. Then F is a **primitive of f on $\mathcal{R}$** if F is analytic on $\mathcal{R}$ and

$$F'(z) = f(z), \quad \text{for all } z \in \mathcal{R}.$$

The function F is also called an **antiderivative** or **indefinite integral** of f on $\mathcal{R}$.

2. **Fundamental Theorem of Calculus**
Let the function f be continuous on $\mathcal{R}$, let F be a primitive of f on $\mathcal{R}$, and let Γ be a contour in $\mathcal{R}$ with initial point α and final point β. Then

$$\int_\Gamma f(z)\,dz = F(\beta) - F(\alpha) = [F(z)]_\alpha^\beta.$$

3. The Chain Rule,

$$g(f(z))f'(z) = (g \circ f)'(z),$$

provides a primitive $g \circ f$ for the integrand in

$$\int_\Gamma g(f(z))f'(z)\,dz.$$

This is a form of integration by substitution.

4. **Contour Independence Theorem**

Let the function f be continuous and have a primitive on a region $\mathcal{R}$, and let Γ_1 and Γ_2 be contours in $\mathcal{R}$ with the same initial point α and the same final point β. Then

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz.$$

5. **Integration by Parts**

Let the functions f and g be analytic on a region $\mathcal{R}$, and let f' and g' be continuous on $\mathcal{R}$. Let Γ be a contour in $\mathcal{R}$ with initial point α and final point β. Then

$$\int_\Gamma f(z)g'(z)\,dz = [f(z)g(z)]_\alpha^\beta - \int_\Gamma f'(z)g(z)\,dz.$$

6. The Fundamental Theorem of Calculus cannot be used to integrate non-differentiable functions.

7. A path or contour Γ is **closed** if its initial and final points coincide.

If Γ is a closed contour, then the value of any contour integral along Γ does not depend on the choice of initial (= final) point.

8. **Closed Contour Theorem**

Let the function f be continuous and have a primitive F on a region $\mathcal{R}$. Then

$$\int_\Gamma f(z)\,dz = 0,$$

for any closed contour Γ in $\mathcal{R}$.

9. A **grid path** is a contour each of whose constituent smooth paths is a line segment parallel to either the real axis or the imaginary axis.

10. **Grid Path Theorem**

If $\mathcal{R}$ is a region, then any two points in $\mathcal{R}$ can be joined by a grid path in $\mathcal{R}$.

11. **Zero Derivative Theorem**

Let the function F be analytic on a region $\mathcal{R}$, and let $F'(z) = 0$, for all z in $\mathcal{R}$. Then F is constant on $\mathcal{R}$.

Section 4: Estimating Contour Integrals

1. Let $\Gamma : \gamma(t)$ $(t \in [a, b])$ be a smooth path. Then the **length of the path Γ**, denoted by $L(\Gamma)$, is

$$L(\Gamma) = \int_a^b |\gamma'(t)|\,dt.$$

The **length of a contour** is the sum of the lengths of its constituent smooth paths.

2. Equivalent smooth paths have the same length. A smooth path and its reverse path have the same length.

3. **Estimation Theorem**

Let f be a function which is continuous on a contour Γ of length L, with

$$|f(z)| \le M, \quad \text{for } z \in \Gamma.$$

Then

$$\left|\int_\Gamma f(z)\,dz\right| \le ML.$$

4. Let $g : [a, b] \longrightarrow \mathbb{C}$ be a continuous function. Then

$$\left|\int_a^b g(t)\,dt\right| \le \int_a^b |g(t)|\,dt.$$

Unit B2 Cauchy's Theorem

Section 1: Cauchy's Theorem

1. A path $\Gamma : \gamma(t)$ $(t \in [a, b])$ is **simple-closed** if it is closed and γ is one-one on $[a, b[$.

   A path $\Gamma : \gamma(t)$ $(t \in [a, b])$ is **simple** if γ is one-one on $[a, b]$.

2. **Jordan Curve Theorem**

   If Γ is a simple-closed path, then the complement $\mathbb{C} - \Gamma$ of Γ is the union of two disjoint regions:

   a bounded region, called the **inside** of Γ, and

   an unbounded region, called the **outside** of Γ.

3. A region $\mathcal{R}$ is **simply-connected** if, whenever Γ is a simple-closed path lying in $\mathcal{R}$, the inside of Γ also lies in $\mathcal{R}$.

   In practice, we usually employ the following *informal definition* when testing a region for simple-connectedness:

   a region is simply-connected if it has no holes in it.

4. **Cauchy's Theorem**

   Let $\mathcal{R}$ be a simply-connected region, and let f be a function which is analytic on $\mathcal{R}$. Then

   $$\int_\Gamma f(z)\,dz = 0,$$

   for any closed contour Γ in $\mathcal{R}$.

5. **Contour Independence Theorem**

   Let $\mathcal{R}$ be a simply-connected region, let f be a function which is analytic on $\mathcal{R}$, and let Γ_1 and Γ_2 be contours in $\mathcal{R}$ with the same initial point α and the same final point β. Then

   $$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz.$$

6. **Convention** Unless otherwise specified, any simple-closed contour in a contour integral will be assumed to be traversed once anticlockwise.

7. **Shrinking Contour Theorem**

   Let $\mathcal{R}$ be a simply-connected region, let Γ be a simple-closed contour in $\mathcal{R}$, let α be a point inside Γ, and let g be a function which is analytic on $\mathcal{R} - \{\alpha\}$. Then

   $$\int_\Gamma g(z)\,dz = \int_C g(z)\,dz,$$

   where C is any circle with centre α, lying inside Γ.

Section 2: The Integral Formula

1. **Cauchy's Integral Formula**

   Let $\mathcal{R}$ be a simply-connected region, let Γ be a simple-closed contour in $\mathcal{R}$, and let f be a function which is analytic on $\mathcal{R}$. Then

   $$f(\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\alpha}\,dz,$$

   for any point α inside Γ.

2. **Liouville's Theorem**

   If f is a bounded entire function, then f is constant.

Section 3: The Derivative Formulas

1. **Cauchy's nth Derivative Formula**

   Let $\mathcal{R}$ be a simply-connected region, let Γ be a simple-closed contour in $\mathcal{R}$, and let f be a function which is analytic on $\mathcal{R}$. Then, for any point α inside Γ, f is n-times differentiable at α and

   $$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-\alpha)^{n+1}}\,dz, \quad \text{for } n = 1, 2, \ldots.$$

2. **Analyticity of Derivatives**

   Let $\mathcal{R}$ be a region, and let f be a function which is analytic on $\mathcal{R}$. Then f possesses derivatives of all orders on $\mathcal{R}$, so that $f', f'', f^{(3)}, \ldots$ are analytic on $\mathcal{R}$.

Section 4: Revision

1. Contour integrals may be evaluated by the following methods:

   parametrization (using the definition of a contour integral) — *Unit B1*,

   Closed Contour Theorem — *Unit B1*,

   Cauchy's Theorem — Section 1,

   Cauchy's Integral Formula — Section 2,

   Cauchy's nth Derivative Formula — Section 3.

2. If the value of a contour integral is known, then evaluating the integral using parametrization yields the value of two real integrals.

3. **Strategy** To evaluate $I = \int_\Gamma \frac{g(z)}{p(z)}\,dz$, where g is analytic on a simply-connected region containing a simple-closed contour Γ, and p is a polynomial function with no zeros on Γ:

   (a) factorize $p(z)$ as $q(z)r(z)$, where the zeros of q lie inside Γ and those of r lie outside Γ; then $f = g/r$ is analytic on $\mathcal{R}$, a simply-connected region which contains Γ;

   (b) expand $1/q(z)$ in partial fractions;

   (c) hence expand $I = \int_\Gamma \frac{f(z)}{q(z)}\,dz$ as a sum of integrals, each of which can be evaluated using Cauchy's Integral Formula and/or Cauchy's nth Derivative Formula.

Section 5: The proof of Cauchy's Theorem

1. **Cauchy's Theorem for a rectangular contour**
Let $\mathcal{R}$ be a simply-connected region, and let f be a function which is analytic on $\mathcal{R}$. Then
$$\int_\Gamma f(z)\,dz = 0,$$
for any rectangular contour Γ in $\mathcal{R}$.
(This theorem is used in the proof of the following theorem.)

2. **Cauchy's Theorem for a closed grid path**
Let $\mathcal{R}$ be a simply-connected region, and let f be a function which is analytic on $\mathcal{R}$. Then
$$\int_\Gamma f(z)\,dz = 0,$$
for any closed grid path Γ in $\mathcal{R}$.
(This theorem is used in the proof of the following theorem.)

3. **Primitive Theorem** (or **Antiderivative Theorem**)
If a function f is analytic on a simply-connected region $\mathcal{R}$, then f has a primitive on $\mathcal{R}$.

4. Cauchy's Theorem (see Section 1) is proved by using the Primitive Theorem and then appealing to the Closed Contour Theorem (*Unit B1*).

5. **Morera's Theorem**
If a function f is continuous on a region $\mathcal{R}$ and $\int_\Gamma f(z)\,dz = 0$, for all rectangular contours Γ in $\mathcal{R}$, then f is analytic on $\mathcal{R}$.

Unit B3 Taylor Series

Section 1: Complex Series

1. The expression $z_1 + z_2 + z_3 + \cdots$ is called an **infinite series** or a **series**. The number z_n is called the **nth term** of the series. The **nth partial sum** of the series is
$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^{n} z_k.$$

2. The series $z_1 + z_2 + z_3 + \cdots$ is **convergent with sum s** if the sequence $\{s_n\}$ of partial sums converges to s. We say that the series **converges** to s, and write
$$z_1 + z_2 + z_3 + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} z_n = s.$$
The series **diverges** if the sequence $\{s_n\}$ diverges.

3. If $\sum_{n=1}^{\infty} z_n$ is a convergent series, then $\{z_n\}$ is a null sequence.

4. **Non-null Test**
If the sequence $\{z_n\}$ is not null, then the series $\sum_{n=1}^{\infty} z_n$ is divergent.

5. The series $\sum_{n=0}^{\infty} az^n = a + az + az^2 + \cdots$, where $a \in \mathbb{C}$, is called a **geometric series with common ratio z**.

6. **Geometric series**

   (a) If $|z| < 1$ and $a \in \mathbb{C}$, then the series $\sum_{n=0}^{\infty} az^n$ is convergent with sum $a/(1-z)$.

   (b) If $|z| \geq 1$ and $a \in \mathbb{C} - \{0\}$, then the series $\sum_{n=0}^{\infty} az^n$ is divergent.

7. The series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is called the **harmonic series**.

8. The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ converges if $p > 1$, and diverges if $p \leq 1$.

9. **Combination Rules**
If $\sum_{n=1}^{\infty} z_n = s$ and $\sum_{n=1}^{\infty} w_n = t$, then

   Sum Rule $\sum_{n=1}^{\infty} (z_n + w_n) = s + t;$

   Multiple Rule $\sum_{n=1}^{\infty} \lambda z_n = \lambda s, \quad \text{for } \lambda \in \mathbb{C}.$

10. The series $\sum_{n=1}^{\infty} z_n$ is convergent if and only if both the series $\sum_{n=1}^{\infty} \operatorname{Re} z_n$ and $\sum_{n=1}^{\infty} \operatorname{Im} z_n$ are convergent. In this case

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \operatorname{Re} z_n + i \sum_{n=1}^{\infty} \operatorname{Im} z_n.$$

11. Comparison Test

If $\sum_{n=1}^{\infty} a_n$ is a real convergent series of positive terms, and if

$$|z_n| \leq a_n, \quad \text{for } n = 1, 2, \ldots,$$

then the series $\sum_{n=1}^{\infty} z_n$ is convergent.

12. The series $\sum_{n=1}^{\infty} z_n$ is **absolutely convergent** if the real series $\sum_{n=1}^{\infty} |z_n|$ is convergent.

13. Absolute Convergence Test

If the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then the series $\sum_{n=1}^{\infty} z_n$ is convergent.

14. Triangle Inequality

If the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then

$$\left| \sum_{n=1}^{\infty} z_n \right| \leq \sum_{n=1}^{\infty} |z_n|.$$

15. Ratio Test

Suppose that $\sum_{n=1}^{\infty} z_n$ is a complex series for which

$$\left| \frac{z_{n+1}}{z_n} \right| \to l \text{ as } n \to \infty.$$

(a) If $0 \leq l < 1$, then $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.

(b) If $l > 1$, then $\sum_{n=1}^{\infty} z_n$ is divergent.

The case $l > 1$ includes the situation where $\left| \frac{z_{n+1}}{z_n} \right| \to \infty$ as $n \to \infty$.

Section 2: Power Series

1. Let $z \in \mathbb{C}$. An expression of the form

$$\sum_{n=0}^{\infty} a_n (z-\alpha)^n = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots,$$

where $\alpha \in \mathbb{C}, a_n \in \mathbb{C}, n = 0, 1, 2, \ldots$, is called a **power series about α.**

2. A power series **converges on a set S** if for each $z \in S$, the corresponding series converges.

3. Let $A = \{z : \sum_{n=0}^{\infty} a_n (z-\alpha)^n \text{ converges}\}$. The function

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n \quad (z \in A)$$

is called the **sum function** of the power series.

4. Radius of Convergence Theorem

For a given power series

$$\sum_{n=0}^{\infty} a_n (z-\alpha)^n = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots,$$

precisely one of the following possibilities occurs:

(a) the series converges only for $z = \alpha$;

(b) the series converges for all z;

(c) there is a number $R > 0$ such that

$\sum_{n=0}^{\infty} a_n (z-\alpha)^n$ converges (absolutely) if $|z-\alpha| < R$,

and

$\sum_{n=0}^{\infty} a_n (z-\alpha)^n$ diverges if $|z-\alpha| > R$.

5. The positive real number R in case (c) of the Radius of Convergence Theorem is called the **radius of convergence** of the power series. Case (a) corresponds to writing $R = 0$, and case (b) to writing $R = \infty$.

6. All the convergence tests of Section 1 can be applied to power series. In particular, the Ratio Test can be used to find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (z-\alpha)^n$ provided that

$$\frac{|a_{n+1}(z-\alpha)^{n+1}|}{|a_n (z-\alpha)^n|}$$

tends to a limit (possibly ∞) as $n \to \infty$. In this case, the radius of convergence is

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

7. Let R be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n.$$

The **disc of convergence** of the power series is the open disc $\{z : |z-\alpha| < R\}$. The disc of convergence is interpreted to be $\varnothing$ if $R = 0$, and to be $\mathbb{C}$ if $R = \infty$.

8. A power series may converge at none, some or all of the points on the boundary of its disc of convergence.

9. **Differentiation Rule**
The power series

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n \quad \text{and} \quad \sum_{n=1}^{\infty} n a_n(z-\alpha)^{n-1}$$

have the same radius of convergence R, say.

Furthermore, if $f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n$, then f is analytic on the disc of convergence $\{z : |z-\alpha| < R\}$, and

$$f'(z) = \sum_{n=1}^{\infty} n a_n(z-\alpha)^{n-1}, \quad \text{for } |z-\alpha| < R.$$

10. **Integration Rule**
The power series

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-\alpha)^{n+1}$$

have the same radius of convergence R, say.

Furthermore, if $f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n$, then the function

$$F(z) = \text{ constant } + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-\alpha)^{n+1}$$

is a primitive of f on $\{z : |z-\alpha| < R\}$.

Section 3: Taylor's Theorem

1. **Taylor's Theorem**
If f is a function which is analytic on the open disc $D = \{z : |z-\alpha| < r\}$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n, \quad \text{for } z \in D. \qquad (*)$$

Moreover, this representation of f is unique, in the sense that if

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n, \quad \text{for } z \in D,$$

then $a_n = f^{(n)}(\alpha)/n!$, for $n = 0, 1, 2, \ldots$.

(In $(*)$, we have employed the conventions $0! = 1, 0^0 = 1$ and $f^{(0)}(z) = f(z)$.)

2. If f is a function with derivatives $f'(\alpha), f''(\alpha), f'''(\alpha), \ldots$ at the point α, then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$$

is called the **Taylor series about α for f**. The coefficient $f^{(n)}(\alpha)/n!$ is known as the **nth Taylor coefficient** (of f at α).

3. If f is an entire function, then the Taylor series about any point α for f converges to $f(z)$ for all $z \in \mathbb{C}$.

4. A function $f : A \longrightarrow \mathbb{C}$ is **even** if

$$f(-z) = f(z), \quad \text{for } z \in A,$$

and **odd** if

$$f(-z) = -f(z), \quad \text{for } z \in A.$$

The Taylor series about 0 of an even (odd) function includes only even (odd) powers of z.

5. **Basic Taylor Series**

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots, \quad \text{for } |z| < 1;$$

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

$$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \quad \text{for } |z| < 1;$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

6. **Binomial series**
If $\alpha \in \mathbb{C}$, then the **binomial series** about 0 for the function $f(z) = (1+z)^\alpha$ is

$$(1+z)^\alpha = 1 + \binom{\alpha}{1} z + \binom{\alpha}{2} z^2 + \binom{\alpha}{3} z^3 + \cdots, \quad \text{for } |z| < 1,$$

where $\binom{\alpha}{n} = \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))}{n!}$.

Section 4: Manipulating Taylor Series

1. The restrictions of the functions tan and sin to the region $S = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$ have inverse functions denoted by $\tan^{-1}$ and $\sin^{-1}$. The domains of both of these inverse functions contain the open unit disc with centre zero, and their derivatives have the following rules:

$$(\tan^{-1})'(z) = \frac{1}{1+z^2}; \quad (\sin^{-1})'(z) = \frac{1}{\sqrt{1-z^2}}.$$

2. **Product Rule**
Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < R,$$

$$g(z) = \sum_{n=0}^{\infty} b_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < R'.$$

If $r = \min\{R, R'\}$, then

$$(fg)(z) = \sum_{n=0}^{\infty} c_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < r,$$

where, for each m, the numbers $c_0, c_1, \ldots, c_m$ are the coefficients of $(z-\alpha)^0, (z-\alpha)^1, \ldots, (z-\alpha)^m$ in

$$\left(\sum_{k=0}^{m} a_k(z-\alpha)^k\right)\left(\sum_{k=0}^{m} b_k(z-\alpha)^k\right).$$

3. **Composition Rule**
Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < R,$$

$$g(w) = \sum_{n=0}^{\infty} b_n(w-\beta)^n, \quad \text{for } |w-\beta| < R'.$$

If $\beta = f(\alpha)$ $(= a_0)$, then, for some $r > 0$,

$$g(f(z)) = \sum_{n=0}^{\infty} c_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < r,$$

where, for each m, the numbers $c_0, c_1, \ldots, c_m$ are the coefficients of $(z-\alpha)^0, (z-\alpha)^1, \ldots, (z-\alpha)^m$ in

$$\sum_{k=0}^{m} b_k\left(\sum_{l=1}^{m} a_l(z-\alpha)^l\right)^k.$$

4. Taylor series may be deduced from known ones (for example, the basic Taylor series about 0 listed in Section 3) by:
   (a) using the Combination Rules (Section 1);
   (b) using the Product Rule;
   (c) using the Composition Rule;
   (d) using the Differentiation Rule (Section 2);
   (e) using the Integration Rule (Section 2);
   (f) using the substitution $w = \lambda z^k$ (where $\lambda \neq 0$ and $k \in \{1, 2, 3, \ldots\}$), for which
   $$|w| < R \iff |z| < \sqrt[k]{R/|\lambda|};$$
   (g) using the substitution $w = z + (\beta - \alpha)$, which changes a Taylor series about β to one about α and for which
   $$|w-\beta| < R \iff |z-\alpha| < R;$$
   (h) using appropriate combinations of methods (a)–(g).

Section 5: The Uniqueness Theorem

1. Let a function f be analytic at α. If
   $$f(\alpha) = f'(\alpha) = f''(\alpha) = \ldots = f^{(k-1)}(\alpha) = 0,$$
   but $f^{(k)}(\alpha) \neq 0$,
   then f has a **zero at α of (finite) order k.**
   A **simple zero** is a zero of order 1.

2. A function f is analytic at a point α, and has a zero of order k at α, if and only if, for some $r > 0$,
   $$f(z) = (z-\alpha)^k g(z), \quad \text{for } |z-\alpha| < r,$$
   where g is analytic at α, and $g(\alpha) \neq 0$.

3. If a function f is analytic on a region $\mathcal{R}$ and not identically zero there, then any zero of f is of finite order.

4. A zero α of a function f is said to be **isolated** if some disc centred at α contains no other zeros of f.

5. **Isolated zeros**
   A zero of finite order is isolated.

6. If a function f is analytic on a region $\mathcal{R}$ and if S is a set of zeros of f with a limit point in $\mathcal{R}$, then f is identically zero on $\mathcal{R}$ (that is, $f(z) = 0$, for all $z \in \mathcal{R}$).

7. **Uniqueness Theorem**
   Let the functions f and g be analytic on a region $\mathcal{R}$ and suppose that f agrees with g throughout a set $S \subseteq \mathcal{R}$, where S has a limit point in $\mathcal{R}$. Then f agrees with g throughout $\mathcal{R}$.

Unit B4 Laurent Series

Section 1: Singularities

1. A function f has an (isolated) **singularity** at the point α if f is analytic on a punctured open disc $\{z : 0 < |z-\alpha| < r\}$, where $r > 0$, but not at α itself.

2. Let f be a function with domain A, and suppose that α is a limit point of A. The function f **tends to infinity as z tends to α** if,

for each sequence $\{z_n\}$ in $A - \{\alpha\}$ such that $z_n \to \alpha$,

$$f(z_n) \to \infty$$

(or, equivalently, if,

for each positive number M, there is a positive number δ such that

$$|f(z)| \geq M, \quad \text{for } z \in A \text{ and } 0 < |z-\alpha| < \delta).$$

We write $f(z) \to \infty$ as $z \to \alpha$.

3. Suppose that a function f has a singularity at α. Then

(a) f has a **removable singularity** at α if there is a function g, analytic at α, and a positive number r, such that

$$f(z) = g(z), \quad \text{for } 0 < |z-\alpha| < r;$$

(b) f has a **pole of order k** at α if there is a function g, analytic at α with $g(\alpha) \neq 0$, and a positive number r, such that

$$f(z) = \frac{g(z)}{(z-\alpha)^k}, \quad \text{for } 0 < |z-\alpha| < r;$$

(c) f has an **essential singularity** at α if the singularity at α is neither removable nor a pole.

In case (a), g is an **analytic extension** of f to $\{z : |z-\alpha| < r\}$.

4. If a function f has a singularity at α and
1. $\lim_{z \to \alpha} f(z)$ does not exist,
2. $f(z)$ does not tend to infinity as $z \to \alpha$,

then α is an essential singularity of f.

Section 2: Laurent's Theorem

1. Let $z \in \mathbb{C}$. An expression of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n = \cdots + \frac{a_{-2}}{(z-\alpha)^2} + \frac{a_{-1}}{(z-\alpha)} + a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots, \qquad (*)$$

where $a_n \in \mathbb{C}$, for $n \in \mathbb{Z}$, is called an **extended power series about α**.

2. For a given z, the extended power series $(*)$ is convergent if $\sum_{n=0}^{\infty} a_n(z-\alpha)^n$, the **analytic part**, and $\sum_{n=1}^{\infty} a_{-n}(z-\alpha)^{-n}$, the **singular part**, are both convergent; its **sum** is found by adding the sums of these two series.

3. Let $A = \{z : \sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n \text{ converges}\}$. The function

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n \quad (z \in A)$$

is called the **sum function** of the extended power series.

4. If the analytic part of an extended power series has disc of convergence $\{z : |z-\alpha| < r\}$ and the singular part converges on $\{z : |z-\alpha| > 1/r'\}$, then the extended power series has **annulus of convergence**

$$A = \{z : |z-\alpha| < r\} \cap \{z : |z-\alpha| > 1/r'\}.$$

Depending on the values of r and r', A may be an open annulus, a punctured open disc, a punctured plane, the 'outside' of a closed disc, or the empty set.

5. **Laurent's Theorem**
If f is a function which is analytic on the open annulus

$$A = \{z : r_1 < |z-\alpha| < r_2\},$$

where $0 \leq r_1 < r_2 \leq \infty$, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n, \quad \text{for } z \in A, \qquad (*)$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{n+1}}\, dz, \quad \text{for } n \in \mathbb{Z},$$

C being any circle with centre α lying in A.
Moreover, the representation $(*)$ of f is unique on A.

6. The representation $(*)$ is the **Laurent series (about α) for the function f on A.**
If A is a punctured open disc, then the representation $(*)$ is called the **Laurent series about α for f.**

7. A function f may have different Laurent series about α on different annuli.

8. If a function f has a singularity at α and if its Laurent series about α is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n,$$

then

(a) f has a removable singularity at α if and only if $a_n = 0$, for all $n < 0$;

(b) f has a pole of order k at α if and only if $a_n = 0$, for all $n < -k$, and $a_{-k} \neq 0$;

(c) f has an essential singularity at α if and only if $a_n \neq 0$, for infinitely many $n < 0$.

9. A Laurent series may be obtained from known Taylor series (for example, by using substitution) and by using partial fractions.

Section 3: Behaviour near a Singularity

1. Let a function f have a singularity at α. Then the following statements are equivalent:

   (A) f has a removable singularity at α;

   (B) $\lim_{z\to\alpha} f(z)$ exists;

   (C) f is bounded on $\{z : 0 < |z-\alpha| < r\}$, for some $r > 0$;

   (D) $\lim_{z\to\alpha} (z-\alpha)f(z) = 0$.

2. Let a function f have a singularity at α. Then the following statements are equivalent:

   (A) f has a pole of order k at α;

   (B) $\lim_{z\to\alpha} (z-\alpha)^k f(z)$ exists, and is non-zero;

   (C) $1/f$ has a removable singularity at α which, when removed, gives rise to a zero of order k at α.

3. Let a function f have a singularity at α. Then f has a pole at α if and only if

   $$f(z) \to \infty \text{ as } z \to \alpha.$$

4. **Casorati–Weierstrass Theorem**

   Suppose that a function f has an essential singularity at α. Let w be any complex number, and let ε and δ be positive real numbers. Then there exists $z \in \mathbb{C}$ such that

   $$0 < |z-\alpha| < \delta \quad \text{and} \quad |f(z) - w| < \varepsilon.$$

Section 4: Evaluating Residues using Laurent Series

1. If a function f is analytic on $D = \{z : 0 < |z-\alpha| < r\}$, then, by Laurent's Theorem,

   $$\int_C \frac{f(w)}{(w-\alpha)^{n+1}}\, dw = 2\pi i a_n,$$

   where $\sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n$ is the Laurent series about α for f, and C is any circle with centre α lying in D.

2. If a function f is analytic on a punctured disc D with centre α, then the coefficient a_{-1} in the Laurent series about α for f is called the **residue of f at α**, and is denoted by $\operatorname{Res}(f, \alpha)$. Hence

   $$\int_C f(z)\, dz = 2\pi i \operatorname{Res}(f, \alpha),$$

   where C is any circle with centre α lying in D.

PART II: INDEX

Absolute Convergence Test 7
analytic extension 10
analytic part 10
Analyticity of Derivatives 5
annulus of convergence 10
antiderivative 3
Antiderivative Theorem 6

Basic Taylor Series 8
binomial series 8

Casorati–Weierstrass Theorem 11
Cauchy's Integral Formula 5
Cauchy's nth Derivative Formula 5
Cauchy's Theorem 5
Closed Contour Theorem 4
Combination Rules for integrals 3
Combination Rules for series 6
Comparison Test 7
Composition Rule for Taylor series 9
contour 3
  closed 4
  final point 3
  initial point 3
Contour Independence Theorem 4, 5

Differentiation Rule 8
disc of convergence 8

Estimation Theorem 4
extended power series 10
  convergent 10

function
  even 8
  odd 8
  tends to ∞ 10

Fundamental Theorem of Calculus 3

geometric series 6
grid path 4
Grid Path Theorem 4

harmonic series 6

indefinite integral 3
inside of a simple-closed path 5
integral along a contour 3
integral along a smooth path 3
Integration by Parts 4
integration by substitution 4
Integration Rule 8
Isolated zeros 9

Jordan Curve Theorem 5

Laurent series 10
Laurent's Theorem 10
length of a contour 4
length of a smooth path 4
Liouville's Theorem 5

Morera's Theorem 6

Non-null Test 6

outside of a simple-closed path 5

parametrizations
  equivalent 3
partial fractions 11
partial sum 6
path
  closed 4
  equivalent 3
  simple 5
  simple-closed 5
pole 10
power series 7
primitive 3
Primitive Theorem 6
Product Rule for Taylor series 9

radius of convergence 7
Radius of Convergence Theorem 7
Ratio Test 7
region
  simply-connected 5
residue 11
reverse contour 3
Reverse Contour Theorem 3
reverse path 3

series 6
Shrinking Contour Theorem 5
singular part 10
singularity 10
  essential 10
  removable 10
sum
  of a series 6
  of an extended power series 10
sum function
  of a power series 7
  of an extended power series 10

Taylor coefficient 8
Taylor series 8
Taylor's Theorem 8
Triangle Inequality 7

Uniqueness Theorem 9

zero
  of finite order 9
  simple 9
Zero Derivative Theorem 4

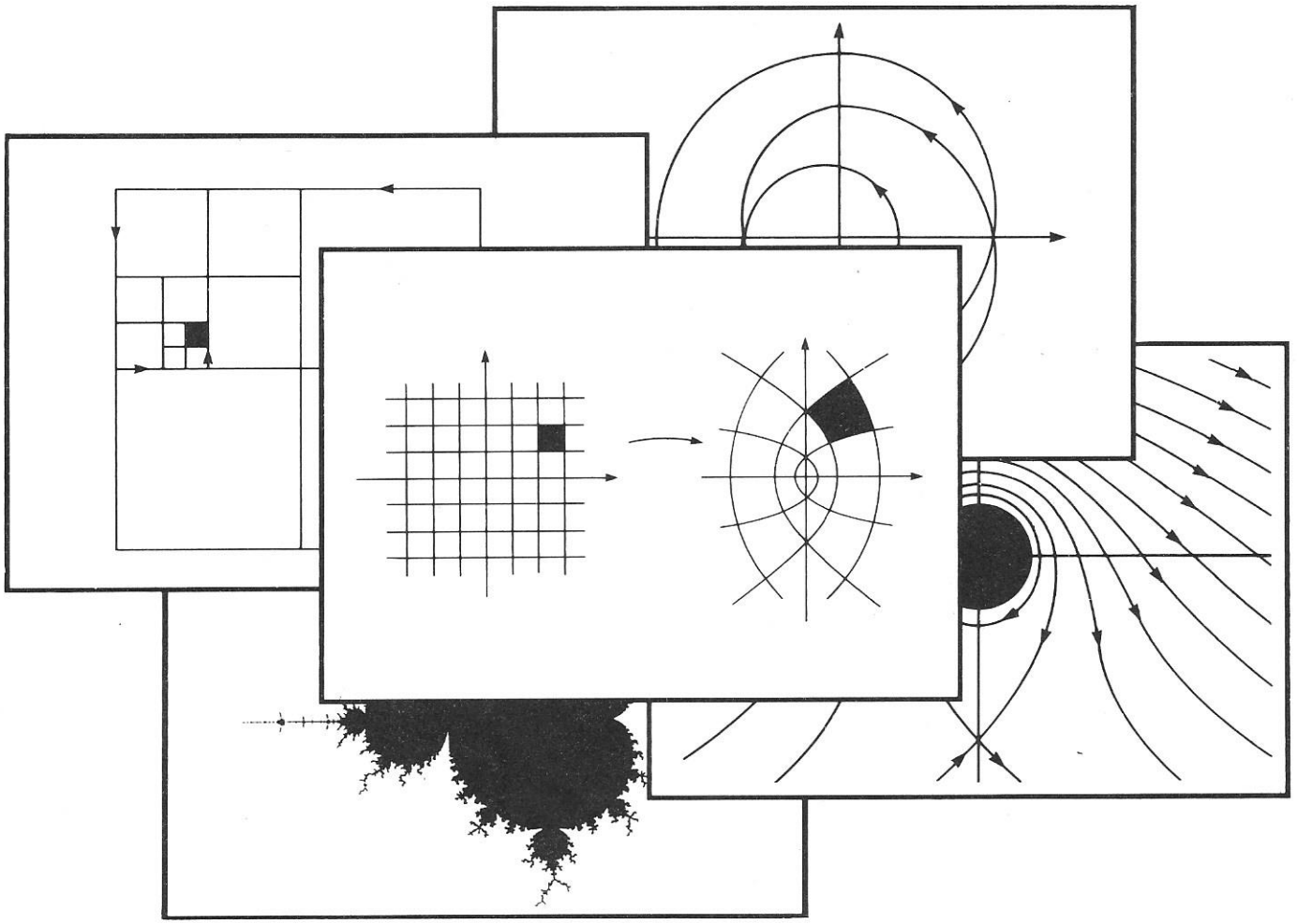
Printed in the United Kingdom by
The Open University

M337 HB3
Mathematics: A Third Level Course

The Open University

COMPLEX ANALYSIS

HANDBOOK THREE



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CONTENTS

Part I: Unit Summaries 3

Unit C1 Residues 3

Unit C2 Zeros and Extrema 5

Unit C3 Analytic Continuation 7

Part II: Index 12

PART I: UNIT SUMMARIES

Unit C1 Residues

Section 1: Calculating Residues

1. Suppose that an analytic function f has a singularity at the point α. Then

$$\mathrm{Res}(f,\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z),$$

provided that this limit exists.

2. g/h **Rule**

Let $f(z) = g(z)/h(z)$, where the functions g and h are analytic at the point α, $h(\alpha) = 0$, and $h'(\alpha) \neq 0$. Then

$$\mathrm{Res}(f,\alpha) = g(\alpha)/h'(\alpha).$$

3. **Cover-up Rule**

Let $f(z) = \dfrac{p(z)}{(z-\alpha)q(z)}$, where the functions p and q are analytic at the point α, and $q(\alpha) \neq 0$. Then

$$\mathrm{Res}(f,\alpha) = p(\alpha)/q(\alpha).$$

4. If a function f has a pole of order k at the point α, then the residue of f at α is given by

$$\mathrm{Res}(f,\alpha) = \frac{1}{(k-1)!}\lim_{z\to\alpha}\left(\frac{d^{k-1}}{dz^{k-1}}((z-\alpha)^k f(z))\right).$$

Section 2: The Residue Theorem

1. **Cauchy's Residue Theorem**

Let $\mathcal{R}$ be a simply-connected region, and let f be a function which is analytic on $\mathcal{R}$ except for a finite number of singularities. Let Γ be any simple-closed contour in $\mathcal{R}$, not passing through any of these singularities. Then

$$\int_\Gamma f(z)\,dz = 2\pi i S,$$

where S is the sum of the residues of f at those singularities that lie inside Γ.

2. **Strategy for evaluating** $\displaystyle\int_0^{2\pi} \Phi(\cos t, \sin t)\,dt$

(a) Replace

$$\cos t \text{ by } \frac{1}{2}\left(z+z^{-1}\right), \sin t \text{ by } \frac{1}{2i}\left(z-z^{-1}\right)$$

and dt by $\dfrac{1}{iz}dz$,

to obtain a contour integral of the form $\int_C f(z)\,dz$ around the unit circle $C = \{z : |z| = 1\}$.

(b) Locate the singularities of the function f lying inside C, and calculate the residues of f at these points.

(c) Evaluate the given integral by calculating

$2\pi i \times$ (the sum of the residues found in step (b)).

Section 3: Evaluating Improper Integrals

1. Let f be a function defined on an unbounded interval $]a,\infty[$. Then the function f has **limit α as r tends to ∞** if

for each sequence $\{r_n\}$ in $]a,\infty[$ such that $r_n \to \infty$,

$$f(r_n) \to \alpha \text{ as } n \to \infty.$$

In this case we write

EITHER $\displaystyle\lim_{r\to\infty} f(r) = \alpha$,

OR $\quad f(r) \to \alpha$ as $r \to \infty$.

2. **Combination Rules**

Let f and g be functions such that

$$\lim_{r\to\infty} f(r) = \alpha \quad \text{and} \quad \lim_{r\to\infty} g(r) = \beta.$$

Then

Sum Rule $\displaystyle\lim_{r\to\infty}(f(r)+g(r)) = \alpha+\beta$;

Multiple Rule $\displaystyle\lim_{r\to\infty}(\lambda f(r)) = \lambda\alpha$, for $\lambda \in \mathbb{C}$;

Product Rule $\displaystyle\lim_{r\to\infty}(f(r)g(r)) = \alpha\beta$;

Quotient Rule $\displaystyle\lim_{r\to\infty}(f(r)/g(r)) = \alpha/\beta$,

provided that $\beta \neq 0$.

3. If p and q are polynomial functions such that the degree of q exceeds the degree of p, then

$$\lim_{r\to\infty}\frac{p(r)}{q(r)} = 0.$$

4. Let f be a continuous function with domain $\mathbb{R}$. Then the **improper integral** $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt$ is

$$\int_{-\infty}^{\infty} f(t)\,dt = \lim_{r\to\infty}\int_{-r}^{r} f(t)\,dt,$$

provided that this limit exists.

Let f be a function which is continuous on the interval $[a,\infty[$. Then the **improper integral** $\displaystyle\int_a^{\infty} f(t)\,dt$ is

$$\int_a^{\infty} f(t)\,dt = \lim_{r\to\infty}\int_a^{r} f(t)\,dt,$$

provided that this limit exists.

5. Let f be a continuous function with domain $\mathbb{R}$. Then:

(a) if f is an odd function, $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt = 0$;

(b) if f is an even function,

$$\int_{-\infty}^{\infty} f(t)\,dt = 2\int_0^{\infty} f(t)\,dt,$$

provided that these integrals exist.

6. Let a function f be defined and continuous at all points of an interval $[a, b]$ except at the point $c \in]a, b[$. Then the **improper integral** $\int_a^b f(t)\,dt$ is

$$\int_a^b f(t)\,dt = \lim_{\varepsilon \to 0} \left(\int_a^{c-\varepsilon} f(t)\,dt + \int_{c+\varepsilon}^b f(t)\,dt \right),$$

where the limit is taken as ε tends to 0 through *positive* values, provided that this limit exists.

Let a function f be continuous at all points of $\mathbb{R}$, except at the point c. Then the **improper integral** $\int_{-\infty}^{\infty} f(t)\,dt$ is

$$\int_{-\infty}^{\infty} f(t)\,dt = \lim_{r \to \infty} \int_{-r}^{r} f(t)\,dt$$

$$= \lim_{r \to \infty} \left(\lim_{\varepsilon \to 0} \left(\int_{-r}^{c-\varepsilon} f(t)\,dt + \int_{c+\varepsilon}^{r} f(t)\,dt \right) \right),$$

provided that these limits exist.

7. Round-the-Pole Lemma

If a function f has a simple pole at α and Γ is the semicircular contour (traversed anticlockwise) from $\alpha + \varepsilon$ to $\alpha - \varepsilon$, then

$$\lim_{\varepsilon \to 0} \int_\Gamma f(z)\,dz = \pi i \operatorname{Res}(f, \alpha).$$

8. Let p and q be polynomial functions such that:

1. the degree of q exceeds that of p by at least two;
2. any poles of p/q on the real axis are simple.

Then

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\,dt = 2\pi i S + \pi i T,$$

where S is the sum of the residues of the function p/q at those poles in the upper half-plane, and T is the sum of the residues of the function p/q at those poles on the real axis.

9. Let p and q be polynomial functions such that:

1. the degree of q exceeds that of p by at least one;
2. any poles of p/q on the real axis are simple.

Then, if $k > 0$,

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} e^{ikt}\,dt = 2\pi i S + \pi i T, \qquad (*)$$

where S is the sum of the residues of the function $z \longmapsto (p(z)/q(z))e^{ikz}$ at those poles in the upper half-plane, and T is the sum of the residues of the function $z \longmapsto (p(z)/q(z))e^{ikz}$ at those poles on the real axis.

By equating the real parts and imaginary parts of Equation $(*)$, we obtain the values of the real improper integrals

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \cos kt\,dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \sin kt\,dt.$$

10. Jordan's Lemma

Let Γ be the semicircular contour (traversed anticlockwise) from r to $-r$ and suppose that a function f is continuous on Γ and satisfies

$$|f(z)| \le M, \quad \text{for } z \in \Gamma.$$

Then, for $k > 0$, we have

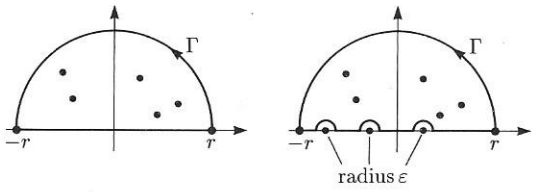
$$\left| \int_\Gamma f(z) e^{ikz} dz \right| \le \frac{M\pi}{k}.$$

11. Improper integrals of the forms

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\,dt, \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \cos kt\,dt \text{ and } \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \sin kt\,dt,$$

which are the subject of the results in items 8 and 9, may also be evaluated by means of a five-step method:

(a) Consider an appropriate contour integral I, where the contour Γ is of one of the following forms.



(b) Evaluate I by using the Residue Theorem.

(c) Split up I into line segments and semicircles.

(d) Estimate the integral along the semicircle of radius r.

(e) Let $r \to \infty$, $\varepsilon \to 0$ (if appropriate) in the equation resulting from step (c), and then deduce the value of the improper integral.

Section 4: Summing Series

1. Let ϕ be an even function which is analytic on $\mathbb{C}$ except for poles at the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ (none of which is an integer), and possibly at 0, and let S_N be the square contour with vertices at $\left(N + \frac{1}{2}\right)(\pm 1 \pm i)$. Suppose also that the function $f(z) = \pi \cot \pi z \cdot \phi(z)$ is such that

$$\lim_{N \to \infty} \int_{S_N} f(z)\,dz = 0.$$

Then

$$\sum_{n=1}^{\infty} \phi(n) = -\frac{1}{2} \left(\operatorname{Res}(f, 0) + \sum_{j=1}^{k} \operatorname{Res}(f, \alpha_j) \right).$$

Further, if ϕ is analytic at 0, then

$$\operatorname{Res}(f, 0) = \phi(0).$$

2. For each $N = 1, 2, \ldots,$

$$|\cot \pi z| \le 2, \quad \text{for } z \in S_N,$$

where S_N is the square contour with vertices at $\left(N + \frac{1}{2}\right)(\pm 1 \pm i)$.

3. Let ϕ be an even function which is analytic on $\mathbb{C}$ except for poles at the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ (none of which is an integer), and possibly at 0, and let S_N be the square contour with vertices at $\left(N + \frac{1}{2}\right)(\pm 1 \pm i)$. Suppose also that the function $f(z) = \pi \operatorname{cosec} \pi z \cdot \phi(z)$ is such that

$$\lim_{N \to \infty} \int_{S_N} f(z)\, dz = 0.$$

Then

$$\sum_{n=1}^{\infty} (-1)^n \phi(n) = -\frac{1}{2}\left(\operatorname{Res}(f, 0) + \sum_{j=1}^{k} \operatorname{Res}(f, \alpha_j) \right).$$

Further, if ϕ is analytic at 0, then

$$\operatorname{Res}(f, 0) = \phi(0).$$

4. For each $N = 1, 2, \ldots,$

$$|\operatorname{cosec} \pi z| \leq 1, \quad \text{for } z \in S_N,$$

where S_N is the square contour with vertices at $\left(N + \frac{1}{2}\right)(\pm 1 \pm i)$.

5. The Laurent series about 0 for cot and cosec are:

$$\cot z = \frac{1}{z} - \frac{1}{3} z - \frac{1}{45} z^3 - \cdots;$$

$$\operatorname{cosec} z = \frac{1}{z} + \frac{1}{6} z + \frac{7}{360} z^3 + \cdots.$$

Unit C2 Zeros and Extrema

Section 1: The Winding Number

1. A **continuous argument function** for a path $\Gamma : \gamma(t)$ $(t \in [a, b])$ lying in $\mathbb{C} - \{0\}$ is a continuous function

$$\theta : [a, b] \longmapsto \mathbb{R}$$

such that, for each $t \in [a, b]$, $\theta(t)$ is an argument of $\gamma(t)$.

Such a function θ satisfies

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{i\theta(t)}, \quad \text{for } t \in [a, b].$$

2. Any path $\Gamma : \gamma(t)$ $(t \in [a, b])$ lying in $\mathbb{C} - \{0\}$ has a continuous argument function θ, which is unique apart from the addition of a constant term of the form $2\pi n$, where $n \in \mathbb{Z}$.

3. The **winding number** of a path $\Gamma : \gamma(t)$ $(t \in [a, b])$ lying in $\mathbb{C} - \{0\}$ **round 0** is

$$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$

where θ is any continuous argument function for Γ.

$\operatorname{Wnd}(\Gamma, 0)$ may be calculated by inspecting a sketch of Γ, if one is available.

4. For $\phi \in \mathbb{R}$, the function Arg_ϕ is defined by

$$\operatorname{Arg}_\phi(z) = \theta \quad (z \in \mathbb{C} - \{0\}),$$

where θ is the argument of z lying in the interval $]\phi - 2\pi, \phi]$.

5. For all $\phi \in \mathbb{R}$, Arg_ϕ is continuous on the cut plane

$$\mathbb{C}_\phi = \left\{ re^{i\theta} : r > 0, \phi - 2\pi < \theta < \phi \right\}.$$

6. For $\phi \in \mathbb{R}$, the function Log_ϕ is defined by

$$\operatorname{Log}_\phi(z) = \log_e |z| + i \operatorname{Arg}_\phi(z) \quad (z \in \mathbb{C} - \{0\}).$$

7. For all $\phi \in \mathbb{R}$, Log_ϕ is analytic on $\mathbb{C}_\phi$ with derivative Log'_ϕ given by

$$\operatorname{Log}'_\phi(z) = 1/z, \quad \text{for } z \in \mathbb{C}_\phi.$$

8. Let $\Gamma : \gamma(t)$ $(t \in [a, b])$ be a closed contour lying in $\mathbb{C} - \{0\}$. Then

$$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z}\, dz.$$

9. The **winding number** of the path $\Gamma : \gamma(t)$ $(t \in [a, b])$ lying in $\mathbb{C} - \{\alpha\}$ **round** α is

$$\operatorname{Wnd}(\Gamma, \alpha) = \frac{1}{2\pi}(\theta_\alpha(b) - \theta_\alpha(a)),$$

where θ_α is a continuous argument function for Γ relative to α (that is, θ_α is continuous on $[a, b]$ and $\theta_\alpha(t)$ is an argument of $\gamma(t) - \alpha$).

10. $\mathrm{Wnd}(\Gamma, \alpha) = \mathrm{Wnd}(\Gamma - \alpha, 0)$,

where

$$\Gamma - \alpha : \gamma(t) - \alpha \quad (t \in [a, b])$$

is the path Γ translated by $-\alpha$.

11. Let Γ be a closed path and let D be an open disc lying in the complement of Γ. Then the function $\alpha \longmapsto \mathrm{Wnd}(\Gamma, \alpha)$ is constant on D.

Section 2: Locating Zeros of Analytic Functions

1. Let an analytic function f have a zero of order n at α. Then the function f'/f has a simple pole at α with

$$\mathrm{Res}(f'/f, \alpha) = n.$$

2. **Argument Principle**

Let a function f be analytic on a simply-connected region $\mathcal{R}$ and let Γ be a simple-closed contour in $\mathcal{R}$, such that $f(z) \neq 0$, for $z \in \Gamma$. Then

$$\mathrm{Wnd}(f(\Gamma), 0) = N,$$

where N is the number of zeros of f inside Γ, counted according to their orders.

3. Let a function f be analytic on a simply-connected region $\mathcal{R}$ and let Γ be a simple-closed contour in $\mathcal{R}$, such that $f(z) \neq \beta$, for $z \in \Gamma$. Then $\mathrm{Wnd}(\Gamma, \beta)$ is the number of zeros of the function $f - \beta$ inside Γ, counted according to their orders.

Unless otherwise stated, zeros are always counted according to their orders.

4. **Rouché's Theorem**

Suppose that

1. the function f is analytic on a simply-connected region $\mathcal{R}$;
2. Γ is a simple-closed contour in $\mathcal{R}$ and
$$|f(z) - g(z)| < |g(z)|, \quad \text{for } z \in \Gamma.$$

Then f has the same number of zeros as g inside Γ. (The function g is known as a 'dominant term' in f on Γ.)

5. **Fundamental Theorem of Algebra**

Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $n \geq 1$ and $a_n \neq 0$. Then p has exactly n zeros, all lying in the open disc $\{z : |z| < R\}$, where

$$R = 1 + \max\{|a_0|/|a_n|, \ldots, |a_{n-1}|/|a_n|\}.$$

Section 3: Local Behaviour of Analytic Functions

1. **Open Mapping Theorem**

Let the function f be analytic and non-constant on a region $\mathcal{R}$ and let G be an open subset of $\mathcal{R}$. Then $f(G)$ is open.

2. If a function f is analytic and non-constant on a region $\mathcal{R}$, then $f(\mathcal{R})$ is a region.

3. Let a function f be analytic on a region $\mathcal{R}$ and let $\alpha \in \mathcal{R}$. Then f is **n-one near α** if there is a region $\mathcal{S}$ in $\mathcal{R}$, with $\alpha \in \mathcal{S}$, and a function ϕ which is analytic and one-one on its domain $\mathcal{S}$, such that

$$f(z) = f(\alpha) + (\phi(z))^n, \quad \text{for } z \in \mathcal{S}.$$

4. **Local Mapping Theorem**

Let a function f be analytic on a region $\mathcal{R}$ and let $\alpha \in \mathcal{R}$. Then the Taylor series about α for f has the form

$$f(z) = f(\alpha) + a_n(z - \alpha)^n + a_{n+1}(z - \alpha)^{n+1} + \cdots, \quad (*)$$

where $n \geq 1$ and $a_n \neq 0$, if and only if f is n-one near α.

5. The Taylor series for f about α takes the form $(*)$ if and only if

$$0 = f'(\alpha) = f''(\alpha) = \cdots = f^{(n-1)}(\alpha),$$

but $f^{(n)}(\alpha) \neq 0$.

6. **Inverse Function Rule**

Let f be a one-one analytic function whose domain is a region $\mathcal{R}$. Then f^{-1} is analytic on $f(\mathcal{R})$ and

$$\left(f^{-1}\right)'(\beta) = \frac{1}{f'(f^{-1}(\beta))}, \quad \text{for } \beta \in f(\mathcal{R}).$$

7. The inverse functions $\tan^{-1}$ and $\sin^{-1}$ are analytic.

8. **Strategy for inverting a Taylor series**

Given the Taylor series about α for f:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n,$$

where $a_1 = f'(\alpha) \neq 0$, we can find the Taylor series about $\beta = f(\alpha)$ for f^{-1}:

$$f^{-1}(w) = \sum_{n=0}^{\infty} b_n (w - \beta)^n,$$

by putting $b_0 = \alpha$ and equating the powers of $(z - \alpha)$ in the identity

$$z - \alpha = b_1(a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots) + b_2(a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots)^2 + \cdots,$$

to obtain equations for $b_1, b_2, \ldots$, in terms of $a_1, a_2, \ldots$.

Section 4: Extreme Values of Analytic Functions

1. Let a function f be defined on a region $\mathcal{R}$. Then the function $|f|$ has a **local maximum** at the point $\alpha \in \mathcal{R}$ if there is some $r > 0$ such that $\{z : |z - \alpha| < r\} \subseteq \mathcal{R}$ and

$$|f(z)| \leq |f(\alpha)|, \quad \text{for } |z - \alpha| < r.$$

2. **Local Maximum Principle**

Let a function f be analytic on a region $\mathcal{R}$. If f is non-constant on $\mathcal{R}$, then the function $|f|$ has no local maxima on $\mathcal{R}$.

3. The **closure** $\overline{A}$ of a set A in $\mathbb{C}$ is

$$\overline{A} = \operatorname{int} A \cup \partial A.$$

4. **Maximum Principle**

Let a function f be analytic on a bounded region $\mathcal{R}$, and continuous on $\overline{\mathcal{R}}$. Then there exists $\alpha \in \partial\mathcal{R}$ such that

$$|f(z)| \leq |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}}.$$

5. **Minimum Principle**

Let a function f be analytic on a bounded region $\mathcal{R}$, and continuous and non-zero on $\overline{\mathcal{R}}$. Then there exists $\alpha \in \partial\mathcal{R}$ such that

$$|f(z)| \geq |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}}.$$

6. **Boundary Uniqueness Theorem**

Let functions f and g be analytic on a bounded region $\mathcal{R}$ and continuous on $\overline{\mathcal{R}}$. If $f = g$ on $\partial\mathcal{R}$, then $f = g$ on $\mathcal{R}$.

7. **Schwarz's Lemma**

Let a function f be analytic on $\{z : |z| < R\}$ with $f(0) = 0$, and suppose that

$$|f(z)| \leq M, \quad \text{for } |z| < R.$$

Then

$$|f(z)| \leq (M/R)|z|, \quad \text{for } |z| < R.$$

Unit C3 Analytic Continuation

Section 1: What is Analytic Continuation?

1. Let f and g be analytic functions whose domains are the regions $\mathcal{R}$ and $\mathcal{S}$, respectively. Then f and g are **direct analytic continuations** of each other if there is a region $\mathcal{T} \subseteq \mathcal{R} \cap \mathcal{S}$ such that

$$f(z) = g(z), \quad \text{for } z \in \mathcal{T}.$$

We also say that g is a **direct analytic continuation of f from $\mathcal{R}$ to $\mathcal{S}$**, and vice versa.

2. Let a function f be continuous on the interval $]0, \infty[$. Then

$$\int_0^\infty f(t)\,dt = \lim_{\epsilon \to 0} \int_\epsilon^1 f(t)\,dt + \lim_{r \to \infty} \int_1^r f(t)\,dt,$$

provided that both these limits exist.

3. Improper integrals of the forms

$$\int_0^\infty \frac{p(t)}{q(t)} \log_e t\,dt, \int_0^\infty \frac{p(t)}{q(t)} t^a\,dt, \int_0^\infty \frac{p(t)}{q(t)}\,dt,$$

where $0 < a < 1$ and p, q are polynomial functions such that p/q is even, the degree of q exceeds that of p by at least two and any poles of p/q on the non-negative real axis are simple, may be evaluated by a five-step method like that given in item 11 of *Unit C1*, Section 3 above.

4. Let p and q be polynomial functions such that

1. the degree of q exceeds the degree of p by at least 2;
2. any poles of p/q on the non-negative real axis are simple.

Then, for $0 < a < 1$,

$$\int_0^\infty \frac{p(t)}{q(t)} t^a dt = -\left(\pi e^{-\pi a i} \operatorname{cosec} \pi a\right) S - (\pi \cot \pi a) T,$$

where S is the sum of the residues of the function

$$f_1(z) = \frac{p(z)}{q(z)} \exp\left(a \operatorname{Log}_{2\pi}(z)\right)$$

in $\mathbb{C}_{2\pi}$, and T is the sum of the residues of the function

$$f_2(z) = \frac{p(z)}{q(z)} \exp\left(a \operatorname{Log} z\right)$$

on the positive real axis.

Section 2: Indirect Analytic Continuation

1. Let an analytic function f have as domain a region $\mathcal{R}$ and let $\alpha \in \mathcal{R}$. If the disc of convergence D of the Taylor series about α for f contains points which are not in $\mathcal{R}$, then the function g, with domain D, defined by the Taylor series is a **direct analytic continuation of f by Taylor series.**

2. For convenience an analytic function f whose domain is a region $\mathcal{R}$ is sometimes written $(f, \mathcal{R})$.

3. The finite sequence of functions

$$(f_1, \mathcal{R}_1), (f_2, \mathcal{R}_2), \ldots, (f_n, \mathcal{R}_n)$$

forms a **chain** if, for $k = 1, 2, \ldots, n-1$,

$(f_{k+1}, \mathcal{R}_{k+1})$ is a direct analytic continuation of $(f_k, \mathcal{R}_k)$.

Any two functions of the chain are called **analytic continuations of each other**, and the chain is said to **join** $(f_1, \mathcal{R}_1)$ to $(f_n, \mathcal{R}_n)$. If $\mathcal{R}_1 = \mathcal{R}_n$, then the chain is said to be **closed.**

Two functions of a chain which are not direct analytic continuations of each other are called **indirect analytic continuations.**

Section 3: Uniform Convergence

1. A sequence of functions $\{f_n\}$ **converges pointwise (to a limit function f) on a set E** if, for each $z \in E$,

$$\lim_{n\to\infty} f_n(z) = f(z).$$

2. A sequence of functions $\{f_n\}$ **converges uniformly (to a limit function f) on a set E** if

for each $\varepsilon > 0$, there is an integer N such that

$$|f_n(z) - f(z)| < \varepsilon, \text{ for all } n > N, \text{ and all } z \in E.$$

We also say that $\{f_n\}$ is **uniformly convergent on E, with limit function f.**

3. **Strategy for proving uniform convergence**

To prove that a sequence of functions $\{f_n\}$ converges uniformly on a set E:

(a) determine the limit function f by evaluating

$$f(z) = \lim_{n\to\infty} f_n(z), \quad \text{for } z \in E;$$

(b) find a null sequence $\{a_n\}$ of positive terms such that

$$|f_n(z) - f(z)| \le a_n, \quad \text{for } n = 1, 2, \ldots, \text{ and all } z \in E.$$

4. If $\{\phi_n\}$ is a sequence of functions, then the series of functions

$$\sum_{n=1}^{\infty} \phi_n = \phi_1 + \phi_2 + \cdots$$

converges pointwise/uniformly on a set E if the sequence of **partial sum functions**

$$f_n(z) = \phi_1(z) + \phi_2(z) + \cdots + \phi_n(z)$$

converges pointwise/uniformly on E, respectively. The limit function f of the sequence $\{f_n\}$ is called the **sum function** of $\sum_{n=1}^{\infty} \phi_n$ **on E**, written

$$f(z) = \sum_{n=1}^{\infty} \phi_n(z) \quad (z \in E).$$

5. **Weierstrass' M-test**

Let $\{\phi_n\}$ be a sequence of functions defined on a set E and suppose that there is a sequence of positive terms $\{M_n\}$, such that

1. $|\phi_n(z)| \le M_n$, for $n = 1, 2, \ldots$, and all $z \in E$;
2. $\sum_{n=1}^{\infty} M_n$ is convergent.

Then the series $\sum_{n=1}^{\infty} \phi_n$ is uniformly convergent on E.

6. **Weierstrass' Theorem**

Let $\{f_n\}$ be a sequence of functions which are analytic on a region $\mathcal{R}$ and which converge uniformly to a function f on each closed disc in $\mathcal{R}$. Then

f is analytic on $\mathcal{R}$

and

the sequence $\{f_n'\}$ converges uniformly to f' on each closed disc in $\mathcal{R}$.

7. When Weierstrass' Theorem is used to prove that a function f defined by a series is analytic, the derivative f' may be obtained by term by term differentiation of the series.

8. The **zeta function** ζ, defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad (\mathrm{Re}\, z > 1),$$

is analytic, and can be continued analytically to $\mathbb{C} - \{1\}$.

Section 4: The Gamma Function

1. Let $\mathcal{R}$ be a region and let K be a complex-valued function of the two variables $z \in \mathcal{R}$ and $t \in [a,b]$, such that

1. K is analytic on $\mathcal{R}$ as a function of z, for each $t \in [a,b]$;
2. K and $\partial K/\partial z$ are continuous on $[a,b]$ as functions of t, for each $z \in \mathcal{R}$;
3. for some $M > 0$,
$$|K(z,t)| \le M, \quad \text{for } z \in \mathcal{R}, t \in [a,b].$$

Then the function
$$f(z) = \int_a^b K(z,t)\,dt \quad (z \in \mathcal{R})$$
is analytic on $\mathcal{R}$ and
$$f'(z) = \int_a^b \frac{\partial K}{\partial z}(z,t)\,dt, \quad \text{for } z \in \mathcal{R}.$$

2. The **gamma function** Γ, defined by
$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt,$$
is such that

(a) Γ is analytic on $H = \{z : \operatorname{Re} z > 1\}$,

(b) $\Gamma(n) = (n-1)!, \quad \text{for } n > 1,$

(c) $\Gamma(z+1) = z\Gamma(z), \quad \text{for } z \in H.$

The identity in part (c) is called the **functional equation of the gamma function.**

3. The gamma function has an analytic continuation Γ to $\mathbb{C} - \{0,-1,-2,\ldots\}$ with simple poles at $0,-1,-2,\ldots$, such that
$$\operatorname{Res}(\Gamma,-k) = \frac{(-1)^k}{k!}, \quad \text{for } k = 0,1,2,\ldots.$$
The functional equation of the gamma function holds on $\mathbb{C} - \{0,-1,-2,\ldots\}$.

4. The value of the gamma function at $\frac{3}{2}$ is
$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}.$$

5. An alternative definition of the gamma function is
$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)},$$
for $z \in \mathbb{C} - \{0,-1,-2,\ldots\}$.

Section 5: Riemann's Legacy

This a reading-only section.

PART II: INDEX

Arg_ϕ 5
Argument Principle 6

Boundary Uniqueness
    Theorem 7

Cauchy's Residue Theorem 3
chain 8
  closed 8
closure of a set 7
Combination Rules for limits
    as $r \to \infty$ 3
continuous argument function 5
Cover-up Rule 3

direct analytic
    continuation 7
direct analytic continuation
    by Taylor series 8

Fundamental Theorem
    of Algebra 6

g/h Rule 3
gamma function 9
  functional equation 9

improper integrals 3, 4
  evaluation of 4, 7
indirect analytic continuation 8
Inverse Function Rule 6

Jordan's Lemma 4

limit α as r tends to ∞ 3
Local Mapping Theorem 6
local maximum 7
Local Maximum Principle 7
Log_ϕ 5

M-Test 8
Maximum Principle 7
Minimum Principle 7

n-one near α 6

Open Mapping Theorem 6

partial sum functions 8
pointwise convergence
  of $\sum_{n=1}^{\infty} \phi_n$ 8
  of $\{f_n\}$ 8

Residue Theorem 3
Rouché's Theorem 6
Round-the-Pole Lemma 4

Schwarz's Lemma 7
sum function 8

Taylor series
  inverting 6

uniform convergence
  of $\sum_{n=1}^{\infty} \phi_n$ 8
  of $\{f_n\}$ 8

Weierstrass' Theorem 8
winding number
  round α 5
  round 0 5

zeta function 8

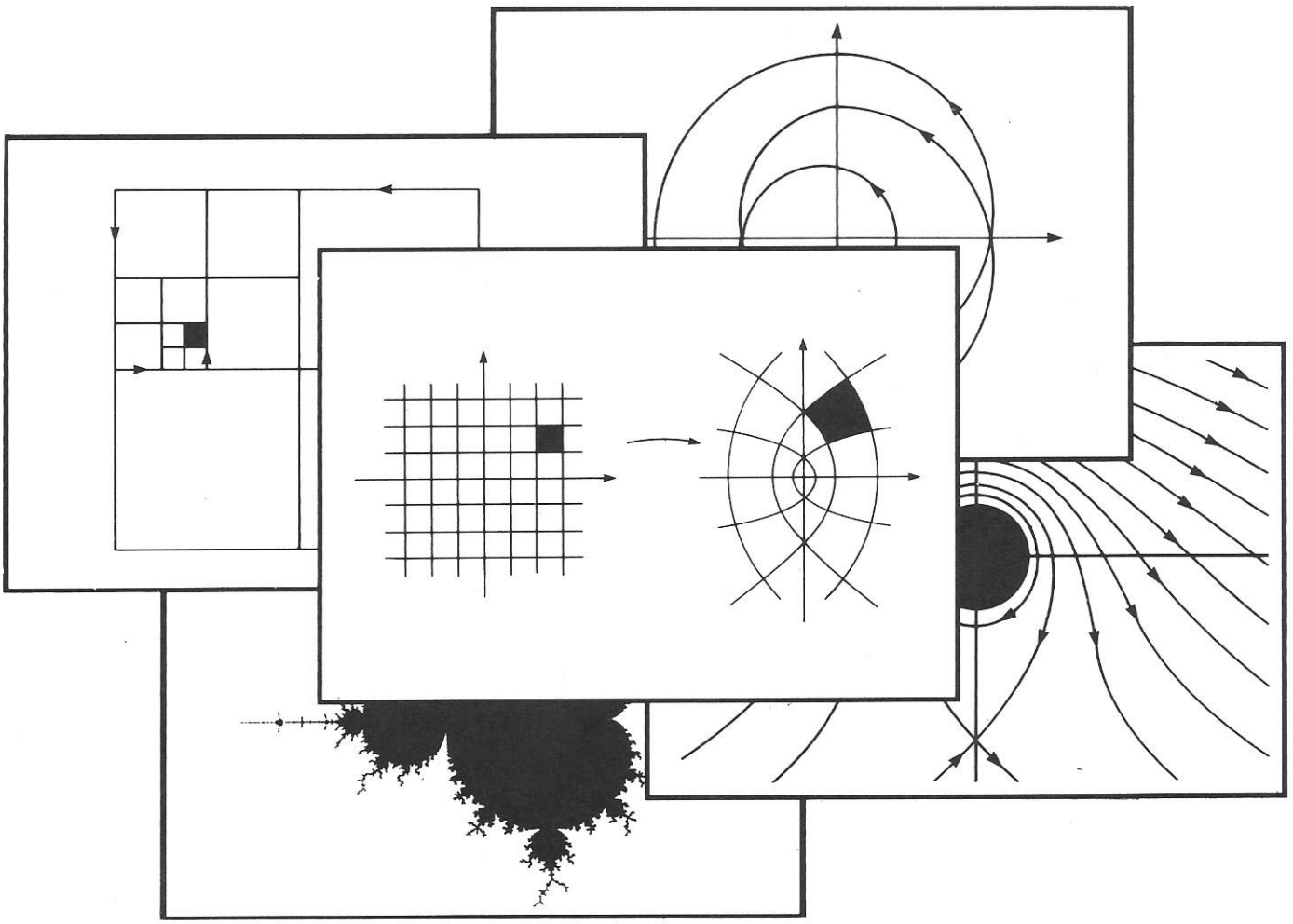
Printed in the United Kingdom by
The Open University

M337 HB4
Mathematics: A Third Level Course

The Open University

COMPLEX ANALYSIS

HANDBOOK FOUR



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CONTENTS

Part I: Unit Summaries 3

Unit D1 Conformal Mappings 3

Unit D2 Fluid Flows 6

Unit D3 The Mandelbrot Set 9

Part II: Index 12

PART I: UNIT SUMMARIES

Unit D1 Conformal Mappings

Section 1: Linear and Reciprocal Functions

1. A function of the form $f(z) = az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$, is called a **linear function.**

2. Linear functions map circles onto circles and straight lines onto straight lines. Furthermore,

(a) given any two circles C_1 and C_2, there is a linear function that maps C_1 onto C_2;

(b) given any two lines L_1 and L_2, there is a linear function that maps L_1 onto L_2.

3. The function $f(z) = 1/z$ is called the **reciprocal function.**

4. **Strategy for finding the equation of the image of a path under $f(z) = 1/z$**

To find the equation of the image $f(\Gamma)$ of a path Γ under $f(z) = 1/z$:

(a) write down an equation that relates the x- and y-coordinates of all points $x + iy$ on Γ;

(b) replace x by $\dfrac{u}{u^2 + v^2}$ and y by $\dfrac{-v}{u^2 + v^2}$;

(c) simplify the resulting equation to obtain an equation that relates the u- and v-coordinates of all points $u + iv$ on the image $f(\Gamma)$.

5. Every line or circle has an equation of the form

$$a\left(x^2 + y^2\right) + bx + cy + d = 0,$$

where $a, b, c, d \in \mathbb{R}$ and $b^2 + c^2 > 4ad$.
Conversely, any such equation represents a line or a circle.

Also

(a) the equation represents a line if and only if $a = 0$;

(b) the line or circle passes through the origin if and only if $d = 0$.

6. The **extended complex plane** $\widehat{\mathbb{C}}$ is the union of the ordinary complex plane $\mathbb{C}$ and one extra point, the **point at infinity**, which is denoted by ∞; thus $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

7. A function f has a **removable singularity**, a **pole of order k**, or an **essential singularity at ∞**, if the function $g(w) = f(1/w)$ has the corresponding type of singularity at 0.

8. Let f be a rational function. The **extended function** $\widehat{f}$ associated with f is the extension of f to $\widehat{\mathbb{C}}$ obtained by defining, for each singularity α of f in $\widehat{\mathbb{C}}$:

$$\widehat{f}(\alpha) = \beta,$$

where β in $\widehat{\mathbb{C}}$ satisfies

$$f(z) \to \beta \text{ as } z \to \alpha.$$

9. If L is a line in $\mathbb{C}$, then $L \cup \{\infty\}$ is an **extended line** in $\widehat{\mathbb{C}}$.

10. A **generalized circle** is either a circle or an extended line. It is completely determined by three distinct points on it.

11. Extended linear functions and the extended reciprocal function map

(a) $\widehat{\mathbb{C}}$ one-one onto $\widehat{\mathbb{C}}$;

(b) generalized circles onto generalized circles.

12. The sphere $\mathbb{S}$ of unit radius and centre the origin is called the **Riemann sphere.**

13. The function $\pi : \mathbb{S} \longrightarrow \widehat{\mathbb{C}}$, defined by

$$\pi(u, v, s) = \begin{cases} \left(\dfrac{u}{1-s}\right) + \left(\dfrac{v}{1-s}\right) i, & (u, v, s) \in \mathbb{S} - \{(0, 0, 1)\}, \\ \infty, & (u, v, s) = (0, 0, 1), \end{cases}$$

is called **stereographic projection.**

14. (a) Under stereographic projection, all ordinary circles on the Riemann sphere $\mathbb{S}$ map to generalized circles in $\widehat{\mathbb{C}}$.

(b) Stereographic projection preserves angles.

Section 2: Möbius Transformations

1. A function of the form

$$f(z) = \frac{az + b}{cz + d}, \quad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0,$$

is called a **Möbius transformation.**

2. Every Möbius transformation is analytic and conformal.

3. The extended function associated with a Möbius transformation is called the **extended Möbius transformation.**

4. Every Möbius transformation is either a linear function, or a composition of linear functions and the reciprocal function.

5. Extended Möbius transformations map

(a) $\widehat{\mathbb{C}}$ one-one onto $\widehat{\mathbb{C}}$;

(b) generalized circles onto generalized circles.

6. **Inverse function of a Möbius transformation**
The Möbius transformation

$$f(z) = \frac{az+b}{cz+d}, \quad \text{where } a,b,c,d \in \mathbb{C} \text{ and } ad-bc \neq 0,$$

is a one-one function from $\mathbb{C} - \{-d/c\}$ onto $\mathbb{C} - \{a/c\}$, with inverse function

$$f^{-1}(w) = \frac{dw-b}{-cw+a}.$$

Furthermore, $\widehat{f}^{-1} = \widehat{f^{-1}}$.

7. The set of extended Möbius transformations has the following group properties.

Closure If $\widehat{f}$ and $\widehat{g}$ are extended Möbius transformations, then so is $\widehat{f} \circ \widehat{g}$.
Identity The identity function on $\widehat{\mathbb{C}}$ is an extended Möbius transformation.
Inverses Every extended Möbius transformation $\widehat{f}$ has an inverse function $\widehat{f}^{-1}$.
Associativity If $\widehat{f}$, $\widehat{g}$ and $\widehat{h}$ are extended Möbius transformations, then

$$\widehat{f} \circ (\widehat{g} \circ \widehat{h}) = (\widehat{f} \circ \widehat{g}) \circ \widehat{h}.$$

8. A point α in the extended complex plane $\widehat{\mathbb{C}}$ is a **fixed point** of an extended Möbius transformation $\widehat{f}$ if $\widehat{f}(\alpha) = \alpha$.

9. An extended Möbius transformation, other than the identity function, has either one or two fixed points in $\widehat{\mathbb{C}}$.

10. If $\widehat{f}$ and $\widehat{g}$ are two extended Möbius transformations that agree on a set consisting of three or more points in $\widehat{\mathbb{C}}$, then $\widehat{f} = \widehat{g}$.

11. Given three distinct points α, β, γ in $\widehat{\mathbb{C}}$, and any other three distinct points α', β', γ' in $\widehat{\mathbb{C}}$, there is a unique extended Möbius transformation $\widehat{f}$ that maps

$$\alpha \text{ to } \alpha', \quad \beta \text{ to } \beta' \quad \text{and} \quad \gamma \text{ to } \gamma'.$$

In particular, if α', β', γ' is the **standard triple of points** 0, 1, ∞, respectively, then $\widehat{f}$ corresponds to

$$f(z) = \frac{(z-\alpha)}{(z-\gamma)}\frac{(\beta-\gamma)}{(\beta-\alpha)}.$$

In general, f is given by the **Implicit Formula**

$$\frac{(z-\alpha)}{(z-\gamma)}\frac{(\beta-\gamma)}{(\beta-\alpha)} = \frac{(w-\alpha')}{(w-\gamma')}\frac{(\beta'-\gamma')}{(\beta'-\alpha')}.$$

12. If C_1 and C_2 are generalized circles, then there is an extended Möbius transformation that maps C_1 onto C_2.

Section 3: Images of Generalized Circles

1. Although the images under an extended Möbius transformation of three distinct points on a generalized circle C do determine the image of C, it may be difficult to identify that image.

2. The **Apollonian form** of the equation of a generalized circle is

$$|z-\alpha| = k|z-\beta|, \quad \text{where } \alpha,\beta \in \mathbb{C} \text{ and } k > 0.$$

If $k = 1$, the generalized circle is an extended line.

3. Let C be a generalized circle. Then α and β are **inverse points with respect to C** if

EITHER α and β are equal and lie on C;
OR there exists an extended Möbius transformation $\widehat{f}$ that maps α to 0, β to ∞, and C to the unit circle.

4. The points α and β in $\widehat{\mathbb{C}}$ are distinct inverse points with respect to a generalized circle C if and only if

EITHER both α and $\beta \in \mathbb{C}$ and C has the equation

$$|z-\alpha| = k|z-\beta|, \quad \text{for some } k > 0;$$

OR one of the points (β say) is ∞ and C has the equation

$$|z-\alpha| = r, \quad \text{for some } r > 0.$$

5. If α is the centre of a circle C, then α and ∞ are inverse points with respect to C.

6. Let $\widehat{f}$ be an extended Möbius transformation. If α and β are inverse points with respect to the generalized circle C, then $\widehat{f}(\alpha)$ and $\widehat{f}(\beta)$ are inverse points with respect to $\widehat{f}(C)$.

7. Let C be the generalized circle with equation

$$|z-\alpha| = k|z-\beta|, \quad \text{where } \alpha,\beta \in \mathbb{C} \text{ and } k > 0.$$

(a) If $k \neq 1$, then C is a circle with centre λ and radius r, where

$$\lambda = \frac{\alpha - k^2\beta}{1-k^2} \quad \text{and} \quad r = \frac{k|\alpha-\beta|}{|1-k^2|}.$$

Also, λ lies on the line through α and β, and

$$(\alpha-\lambda)(\overline{\beta-\lambda}) = r^2. \qquad (*)$$

(b) If $k = 1$, then C is the line through $\frac{1}{2}(\alpha+\beta)$ perpendicular to $\beta - \alpha$.

8. The process of applying the function $f(z) = 1/\overline{z}$ is known as **inversion**.

9. Let C be a generalized circle and let β be an arbitrary point of $\widehat{\mathbb{C}}$. Then β has a unique inverse point α with respect to C, which can be found by applying Equation $(*)$ if C is a circle or by reflection if C is an extended line.

Section 4: Transforming Regions

1. An **open disc centred at ∞** is a set of the form
$\{z : |z| > M\} \cup \{\infty\}$.

2. Let A be a subset of $\widehat{\mathbb{C}}$ and let $\alpha \in \widehat{\mathbb{C}}$. Then α is a **boundary point in $\widehat{\mathbb{C}}$ of A** if each open disc centred at α contains at least one point of A and at least one point of $\widehat{\mathbb{C}} - A$.

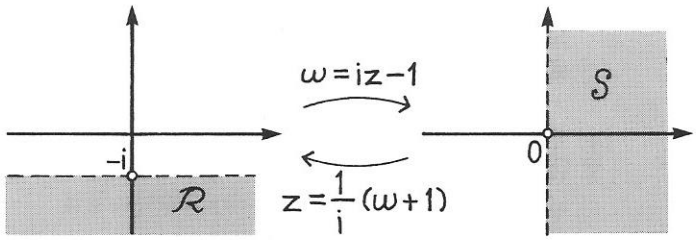
The set of boundary points in $\widehat{\mathbb{C}}$ of A forms the **boundary in $\widehat{\mathbb{C}}$ of A**.

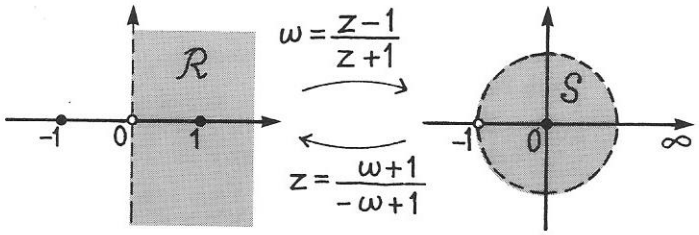
3. Let f be a Möbius transformation and let $\mathcal{R}$ be a region in the domain of f. Then $f(\mathcal{R})$ is a region and the extended Möbius transformation $\widehat{f}$ maps the boundary in $\widehat{\mathbb{C}}$ of $\mathcal{R}$ onto the boundary in $\widehat{\mathbb{C}}$ of $f(\mathcal{R})$.

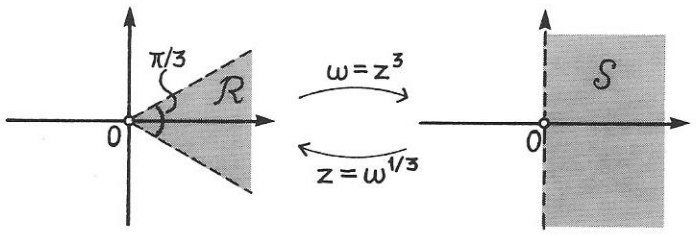
4. Any generalized circle divides $\widehat{\mathbb{C}}$ into two parts, each of which is called a **generalized open disc.**

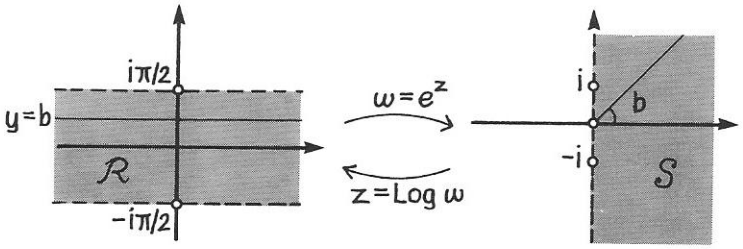
5. Conformal mappings between basic regions may be used to find, by composition, conformal mappings between other regions.

Examples of one-one conformal mappings from $\mathcal{R}$ onto $\mathcal{S}$ and their inverse functions are given in the following figures.









6. The restriction of the function tan to
$\mathcal{R} = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$
has inverse function $\tan^{-1}$ defined by
$$\tan^{-1} w = \frac{1}{2i} \operatorname{Log}\left(\frac{1+iw}{1-iw}\right) \quad (w \in \mathcal{S}),$$
where $\mathcal{S} = \mathbb{C} - \{iv : |v| \geq 1\}$.

7. The restriction of the function sin to
$\mathcal{R} = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$
has inverse function $\sin^{-1}$ defined by
$$\sin^{-1} w = \frac{1}{i} \operatorname{Log}(iw + \sqrt{1-w^2}) \quad (w \in \mathcal{S}),$$
where $\mathcal{S} = \mathbb{C} - \{u \in \mathbb{R} : |u| \geq 1\}$.

8. The **Joukowski function** J is defined by
$$J(z) = z + \frac{1}{z} \quad (z \neq 0).$$
The restriction of J to $\{z : |z| > 1\}$ has inverse function J^{-1} defined by
$$J^{-1}(w) = \tfrac{1}{2}\left(w + w\sqrt{1 - 4/w^2}\right) \quad (w \in \mathbb{C} - [-2, 2]).$$

Section 5: The Riemann Mapping Theorem

This is a reading-only section.

Unit D2 Fluid Flows

Section 1: Setting up the Model

1. Basic Mathematical Model

We assume that

(i) the fluid forms a continuum, and any spatial variation of the flow velocity is continuous;

(ii) the flow is two-dimensional;

(iii) the flow is steady.

With these assumptions, we may represent the **flow velocity** at all times by a continuous complex function q, whose domain is the region occupied by the fluid.

2. If $q(z_0) = 0$, then z_0 is a **stagnation point** of the flow with velocity function q.

3. If q is a constant function, then the associated flow is a **uniform flow**.

4. A **streamline** through the point z_0, for a flow with velocity function q, is a smooth path $\Gamma : \gamma(t)$ $(t \in I)$ such that

(a) $\gamma'(t) = q(\gamma(t))$, for $t \in I$;

(b) $z_0 = \gamma(t_0)$, for some $t_0 \in I$.

If $q(z_0) = 0$, then $\{z_0\}$ is a **degenerate streamline**, with constant parametrization

$$\gamma(t) = z_0 \quad (t \in I).$$

5. The component of $q(z)$ in the direction specified by $e^{i\theta}$ is

$$q_\theta(z) = \mathrm{Re}\left(\overline{q(z)}e^{i\theta}\right).$$

The component of $q(z)$ in the direction specified by $e^{i(\theta-\pi/2)}$ is

$$q_{(\theta-\pi/2)}(z) = \mathrm{Im}\left(\overline{q(z)}e^{i\theta}\right).$$

6. The conjugate velocity function $\overline{q}$ is given by

$$\overline{q}(z) = \overline{q(z)}.$$

7. If $\Gamma : \gamma_1(t)$ $(t \in [a, b])$ is a smooth path of length L, then there is an equivalent parametrization $\gamma(s)$ $(s \in [0, L])$ such that

$$|\gamma'(s)| = 1, \quad \text{for } 0 \le s \le L.$$

Such an equivalent parametrization is called a **unit-speed parametrization**.

8. If $\Gamma : \gamma(s)$ $(s \in [0, L])$ is a smooth path with unit-speed parametrization, which lies in the region $\mathcal{R}$ of a flow velocity function q, then, for each $s \in [0, L]$, the flow velocity $q(\gamma(s))$ has

(a) **tangential component** $q_T(s)$ in the direction specified by $\gamma'(s)$;

(b) **normal component** $q_N(s)$ in the direction specified by $-i\gamma'(s)$.

9. The **circulation** of q along Γ is

$$\mathcal{C}_\Gamma = \int_0^L q_T(s)\,ds,$$

and the **flux** of q across Γ is

$$\mathcal{F}_\Gamma = \int_0^L q_N(s)\,ds.$$

10. $$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \overline{q}(z)\,dz.$$

11. A velocity function with domain a region $\mathcal{R}$ is

(a) **locally circulation-free** if $\mathcal{C}_\Gamma = 0$ for each simple-closed contour Γ in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$;

(b) **locally flux-free** if $\mathcal{F}_\Gamma = 0$ for each simple-closed contour in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$.

12. A **model flow** of a fluid is described by a continuous complex velocity function (whose domain is a region) which is locally circulation-free and locally flux-free.

13. A fluid flow is a model flow on a region $\mathcal{R}$ if and only if

$$\int_\Gamma \overline{q}(z)\,dz = 0,$$

for each simple-closed contour Γ in $\mathcal{R}$ whose inside also lies in $\mathcal{R}$.

14. A steady two-dimensional fluid flow with continuous velocity function q on a region $\mathcal{R}$ is a model flow if and only if its conjugate velocity function $\overline{q}$ is analytic on $\mathcal{R}$.

15. Let q be a model flow velocity function with domain a region $\mathcal{R}$, and let D be a punctured open disc in $\mathcal{R}$ with centre α. Then

(a) α is a **source of strength** $\mathcal{F}$ if $\mathcal{F}_\Gamma = \mathcal{F} \neq 0$ for each simple-closed contour Γ in D which surrounds α;

(b) α is a **vortex of strength** $\mathcal{C}$ if $\mathcal{C}_\Gamma = \mathcal{C} \neq 0$ for each simple-closed contour Γ in D which surrounds α.

A source of negative strength is a **sink**.

16. Let q be a continuous complex velocity function on a region $\mathcal{R}$, and suppose that $q_1 = \mathrm{Re}\, q$ and $q_2 = \mathrm{Im}\, q$ have partial derivatives with respect to x and y which are continuous. Then q is

(a) locally circulation-free if and only if

$$\frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} = 0 \text{ on } \mathcal{R};$$

(b) locally flux-free if and only if

$$\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0 \text{ on } \mathcal{R}.$$

Section 2: Complex Potential Functions

1. Let q be a model flow velocity function with domain $\mathcal{R}$. A function Ω which is a primitive of $\overline{q}$ is called a **complex potential function** for the flow.

Moreover,

$$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \Omega'(z)\,dz = \Omega(\beta) - \Omega(\alpha),$$

where Γ is any contour lying in the domain of Ω with initial and final points α and β.

Such a complex potential function Ω always exists on a simply-connected subregion of $\mathcal{R}$, by the Primitive Theorem.

2. If Ω is a complex potential function on a simply-connected subregion $\mathcal{S}$ of the flow region $\mathcal{R}$, then

$\mathrm{Im}(\Omega(z))$ is constant along each streamline within $\mathcal{S}$.

3. Each point of the flow region has just one streamline through it.

4. If $\Omega(z) = \Phi(x,y) + i\Psi(x,y)$, then $\Psi = \mathrm{Im}\,\Omega$ is the **stream function** for the model flow with complex potential function Ω and the family of curves given by

$\Psi(x,y) = \text{constant}$

forms the steamlines of the flow.

5. There is no flux across a streamline, and so a streamline of given flow can act as a boundary of another flow.

Section 3: Flow past an Obstacle

1. Some standard notation:

$K_a = \{z : |z| \le a\}$;

$\Omega_{a,c}(z) = z + \dfrac{a^2}{z} - ic\,\mathrm{Log}\,z$;

$q_{a,c}(z) = \overline{\Omega'_{a,c}(z)} = \overline{1 - \dfrac{a^2}{z^2} - \dfrac{ic}{z}}$;

where $a, c \in \mathbb{R}$, with $a > 0$.

2. For $a > 0$, $c \in \mathbb{R}$, the model flow velocity function $q_{a,c}$ satisfies the following properties:

(a) $\lim\limits_{z\to\infty} q_{a,c}(z) = 1$;

(b) ∂K_a is made up of streamlines for $q_{a,c}$;

(c) for any simple-closed contour surrounding K_a,

(i) $\mathcal{C}_\Gamma = \mathrm{Re}\displaystyle\int_\Gamma \overline{q_{a,c}}(z)\,dz = 2\pi c$,

and

(ii) $\mathcal{F}_\Gamma = \mathrm{Im}\displaystyle\int_\Gamma \overline{q_{a,c}}(z)\,dz = 0$.

3. An **obstacle** is a compact connected set K in $\mathbb{C}$, such that $\mathbb{C} - K$ is also connected.

4. **Obstacle Problem**

Given an obstacle K and a real number c, we seek a model flow velocity function q with domain the region $\mathcal{R} = \mathbb{C} - K$, satisfying the following properties:

(a) $\lim\limits_{z\to\infty} q(z) = 1$;

(b) there is a complex potential function Ω for q on either $\mathcal{R}$ or $\mathcal{R} - \Sigma$, where Σ is a simple smooth path in $\mathcal{R}$, and a real constant k such that

for each $\alpha \in \partial K$, we have $\lim\limits_{z\to\alpha} \mathrm{Im}(\Omega(z)) = k$;

(c) for any simple-closed contour Γ in $\mathcal{R}$ surrounding K,

$\mathcal{C}_\Gamma = 2\pi c$.

5. For the flow in the Obstacle Problem, $2\pi c$ is called the **circulation around the obstacle** K.

6. The model flow velocity function $q_{a,c}$ solves the Obstacle Problem for $K = K_a$, with circulation $2\pi c$ around K.

7. **Flow Mapping Theorem**

Let K be an obstacle and let f be a one-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_a$, where $a > 0$, such that

$$f(z) = z + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots, \quad \text{for } |z| > R,$$

where $R > 0$ and $a_0, a_{-1}, a_{-2}, \ldots \in \mathbb{C}$. Then the velocity function

$$q(z) = q_{a,c}(f(z))\overline{f'(z)} \quad (z \in \mathbb{C} - K)$$

is the unique solution to the Obstacle Problem for K, with complex potential function

$\Omega = \Omega_{a,c} \circ f$.

8. The function

$$J_a(z) = z + \frac{a^2}{z} \quad (z \in \mathbb{C} - \{0\}),$$

where $a > 0$, has the inverse function

$$J_a^{-1}(w) = \tfrac{1}{2}\left(w + w\sqrt{1 - 4a^2/w^2}\right) \qquad (w \in \mathbb{C} - [-2a, 2a]).$$

9. A point at which a function f has zero derivative is a **critical point** of f.

10. A **Joukowski aerofoil** (in the z-plane) is an obstacle which (possibly after an appropriate translation or rotation) has boundary $J_a(B)$, where B is a circle which passes through the critical point $w = a$ of the function $J_a(w) = w + a^2/w$ and surrounds the other critical point $w = -a$.

The point $z = 2a$ is called the **trailing edge** of the aerofoil.

11. A symmetric aerofoil in a uniform stream in the direction of the positive x-axis has **angle of attack** ϕ, if ϕ is the angle measured clockwise from the negative x-axis to the line of symmetry of the aerofoil.

12. The restriction of the function

$$J_\alpha(z) = z + \frac{\alpha^2}{z} \quad (z \in \mathbb{C} - \{0\}),$$

where $\alpha \in \mathbb{C}$, $\alpha \neq 0$, to $\mathbb{C} - K_{|\alpha|}$ is one-one and has inverse function

$$J_\alpha^{-1}(w) = \tfrac{1}{2}\left(w + w\sqrt{1 - 4\alpha^2/w^2}\right) \quad (w \in \mathbb{C} - L(-2\alpha, 2\alpha)),$$

where $L(-2\alpha, 2\alpha)$ is the line segment between -2α and 2α.

13. The following identity is often useful:

$$J_\alpha^{-1}(w) + \alpha^2 / J_\alpha^{-1}(w) = w.$$

Section 4: Lift and Drag

1. In a fluid of constant density ρ, the relationship between fluid pressure $p(z)$ and fluid speed $|q(z)|$ is given by **Bernoulli's Equation**

$$p(z) + \tfrac{1}{2}\rho|q(z)|^2 = p_0,$$

where the constant p_0 is the fluid pressure at any stagnation point.

2. The **total force** F acting on an obstacle K whose boundary is a simple-closed smooth path of length L is

$$F = i\int_0^L p(\gamma(s))\gamma'(s)\,ds,$$

where $\gamma\colon [0, L] \longrightarrow \partial K$ is a unit-speed parametrization of ∂K.

3. If F is the total force acting on an obstacle, due to a fluid whose velocity function satisfies the Obstacle Problem, then

(a) $\operatorname{Re} F$ is called the **drag** on the obstacle;

(b) $\operatorname{Im} F$ is called the **lift** on the obstacle.

4. **Blasius' Theorem**

Let q be a solution to the Obstacle Problem for an obstacle K, whose boundary ∂K is a simple-closed smooth path, and suppose that q has a continuous extension to ∂K. Then the conjugate of the total force acting on K is

$$\overline{F} = \tfrac{1}{2}i\rho\int_{\partial K}(\overline{q}(z))^2\,dz,$$

where ρ is the density of the fluid.

5. If K is an obstacle whose boundary is not a simple-closed smooth path, then the conjugate of the total force acting on K is

$$\overline{F} = \tfrac{1}{2}i\rho\int_{\Gamma}(\overline{q}(z))^2\,dz,$$

where Γ is any simple-closed contour surrounding K.

6. **Kutta-Joukowski Lift Theorem**

Let q be a solution to the Obstacle Problem for an obstacle K, with circulation $2\pi c$ around K. Then the obstacle experiences a total force

$$F = -2\pi c\rho i.$$

7. **Kutta-Joukowski Hypothesis**

The circulation around a Joukowski aerofoil is such that the flow velocity q is bounded throughout the flow region, and $q(z)$ tends to a limit as z approaches the trailing edge of the aerofoil.

Unit D3 The Mandelbrot Set

Section 1: Iteration of Analytic Functions

1. A sequence $\{z_n\}$ defined by

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots,$$

where f is a function, is called an **iteration sequence** with **initial term** z_0.

2. The **nth iterate** of a function f is the function obtained by applying the function f exactly n times:

$$f^n = f \circ f \circ \cdots \circ f.$$

Also, f^0 denotes the identity function $f^0(z) = z$.

3. A point α is a **fixed point** of a function f if $f(\alpha) = \alpha$.

The equation $f(z) = z$ is called the **fixed point equation**.

4. Let α be a fixed point of an analytic function f and suppose that $|f'(\alpha)| < 1$. Then there exists $r > 0$ such that

$$\lim_{n\to\infty} f^n(z_0) = \alpha, \quad \text{for } |z_0 - \alpha| < r.$$

5. The fixed point α of an analytic function f is

(a) **attracting**, if $|f'(\alpha)| < 1$;

(b) **repelling**, if $|f'(\alpha)| > 1$;

(c) **indifferent**, if $|f'(\alpha)| = 1$;

(d) **super-attracting**, if $f'(\alpha) = 0$.

6. If α is an attracting fixed point of an analytic function f, then the **basin of attraction** of α under f is the set

$$\{z : f^n(z) \to \alpha \text{ as } n \to \infty\}.$$

7. The functions f and g are **conjugate** to each other if

$$g = h \circ f \circ h^{-1},$$

for some one-one function h called the **conjugating function**. If the sequence $\{z_n\}$ is defined by

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots,$$

for some z_0, and $w_n = h(z_n)$, for $n = 0, 1, 2, \ldots$, then the sequence $\{w_n\}$ satisfies

$$w_{n+1} = g(w_n), \quad \text{for } n = 0, 1, 2, \ldots,$$

and $\{z_n\}$ and $\{w_n\}$ are called **conjugate iteration sequences**.

8. Let p be a polynomial function. Then $\{z_n\}$, where

$$z_{n+1} = z_n - \frac{p(z_n)}{p'(z_n)}, \quad n = 0, 1, 2, \ldots,$$

is the **Newton–Raphson iteration sequence**, and the function N, where

$$N(z) = z - \frac{p(z)}{p'(z)},$$

is the **Newton–Raphson function** corresponding to p.

Section 2: Iterating Complex Quadratics

1. The iteration sequence

$$z_{n+1} = az_n^2 + bz_n + c, \quad n = 0, 1, 2, \ldots,$$

where $a \neq 0$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$

where $d = ac + \frac{1}{2}b - \frac{1}{4}b^2$. The conjugating function is $h(z) = az + \frac{1}{2}b$.

2. The set of functions $\{P_c : c \in \mathbb{C}\}$ defined by

$$P_c(z) = z^2 + c,$$

where $c \in \mathbb{C}$, is the **family of basic quadratic functions**.

3. Let $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$. Then, for $|z_0| > r_c$,

$\{|P_c^n(z_0)|\}$ is an increasing sequence,

and

$$P_c^n(z_0) \to \infty \text{ as } n \to \infty.$$

4. For $c \in \mathbb{C}$, the **escape set** E_c is

$$E_c = \{z : P_c^n(z) \to \infty \text{ as } n \to \infty\}.$$

The **keep set** K_c is the complement of E_c.

5. A set A is **completely invariant** under a function f if

$$z \in A \iff f(z) \in A.$$

6. For each $c \in \mathbb{C}$, the escape set E_c and the keep set K_c have the following properties:

(a) $E_c \supseteq \{z : |z| > r_c\}$ and $K_c \subseteq \{z : |z| \leq r_c\}$;

(b) E_c is open and K_c is closed;

(c) $E_c \neq \mathbb{C}$ and $K_c \neq \varnothing$;

(d) E_c and K_c are each completely invariant under P_c;

(e) E_c and K_c are each symmetric under rotation by π about 0;

(f) E_c is connected and K_c has no holes in it.

7. The point α is a **periodic point**, with period p, of a function f if

$$f^p(\alpha) = \alpha, \text{ but } f^k(\alpha) \neq \alpha, \text{ for } k = 1, 2, \ldots, p-1.$$

The p points

$$\alpha,\ f(\alpha),\ f^2(\alpha),\ \ldots,\ f^{p-1}(\alpha)$$

then form a **cycle of period p**, or a **p-cycle** of f.

All the periodic points of P_c lie in the keep set K_c.

8. Let α, $f(\alpha)$, $f^2(\alpha)$, ..., $f^{p-1}(\alpha)$ form a p-cycle of an analytic function f. Then

(a) $$(f^p)'(\alpha) = f'(\alpha) \times f'(f(\alpha)) \times f'\big(f^2(\alpha)\big) \times \cdots \times f'\big(f^{p-1}(\alpha)\big)$$

and, moreover,

(b) the derivative of f^p takes the same value at each point of the p-cycle; that is,

$$(f^p)'(\alpha) = (f^p)'(f(\alpha)) = (f^p)'(f^2(\alpha)) = \ldots = (f^p)'\big(f^{p-1}(\alpha)\big).$$

9. Let α belong to a p-cycle of an analytic function f. The number $(f^p)'(\alpha)$ is the **multiplier** of the p-cycle.

10. If α is a periodic point, with period p, of an analytic function f, then α and the corresponding p-cycle are

(a) **attracting**, if $\left|(f^p)'(\alpha)\right| < 1$;

(b) **repelling**, if $\left|(f^p)'(\alpha)\right| > 1$;

(c) **indifferent**, if $\left|(f^p)'(\alpha)\right| = 1$;

(d) **super-attracting**, if $(f^p)'(\alpha) = 0$.

11. Let α be a periodic point of the function P_c.

(a) If α is attracting, then α is an interior point of K_c.

(b) If α is repelling, then α is a boundary point of K_c.

12. The **Julia set** J_c of P_c is the boundary of K_c. K_c is called the 'filled-in Julia set'.

Section 3: Graphical Iteration

1. **Graphical iteration** with a real function f is the process of constructing the sequence $\{x_n\}$, where

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots,$$

for a given x_0, by drawing alternately vertical and horizontal lines joining the points

$(x_0, 0)$ to (x_0, x_1) $\quad (= (x_0, f(x_0)))$

(x_0, x_1) to (x_1, x_1)

(x_1, x_1) to (x_1, x_2) $\quad (= (x_1, f(x_1)))$

(x_1, x_2) to (x_2, x_2)

(x_2, x_2) to (x_2, x_3) $\quad (= (x_2, f(x_2)))$

and so on,

using the graphs $y = f(x)$ and $y = x$.

2. If $c \in \mathbb{R}$, then the real function P_c has

(a) no real fixed points if $c > \frac{1}{4}$;

(b) the single fixed point $\frac{1}{2}$, if $c = \frac{1}{4}$;

(c) the two real fixed points $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$, if $c < \frac{1}{4}$.

3. If $c > \frac{1}{4}$, then $K_c \cap \mathbb{R} = \varnothing$.

4. If $-2 \le c \le \frac{1}{4}$, then $K_c \cap \mathbb{R} = I_c$, where

$$I_c = \left[-\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - c},\ \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c}\right].$$

5. If $c < -2$, then the set $K_c \cap \mathbb{R}$ consists of the closed interval I_c from which a sequence of disjoint, non-empty, open subintervals of I_c has been removed. In particular, $0 \notin K_c$.

Section 4: The Mandelbrot Set

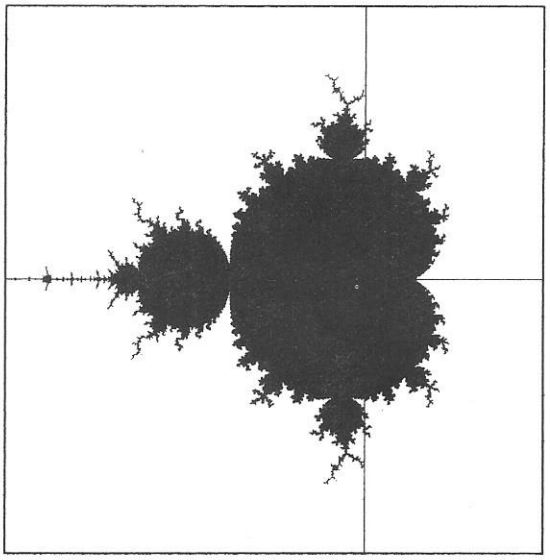
1. A set A is **disconnected** if there are disjoint open sets G_1 and G_2 such that

$$A \cap G_1 \neq \varnothing, \quad A \cap G_2 \neq \varnothing \quad \text{and} \quad A \subseteq G_1 \cup G_2.$$

A set A is **connected** if it is not disconnected.

2. Any pathwise connected set is connected.

3. The **Mandelbrot set** is the set M of complex numbers c such that K_c is connected.



The square represented in this figure is

$$\{c : -2 \le \operatorname{Re} c \le 1, -1.5 \le \operatorname{Im} c \le 1.5\}.$$

4. For any $c \in \mathbb{C}$,

$$K_c \text{ is connected} \iff 0 \in K_c.$$

5. The Mandelbrot set M can be specified as follows:

$$M = \{c : |P_c^n(0)| \le 2, \text{ for } n = 1, 2, \ldots\}.$$

6. The Mandelbrot set M

(a) is a compact subset of $\{c : |c| \le 2\}$;

(b) is symmetric under reflection in the real axis;

(c) meets the real axis in the interval $\left[-2, \frac{1}{4}\right]$;

(d) has no holes in it.

7. The Mandelbrot set is connected.

8. If the function P_c has an attracting cycle, then $c \in M$.

9. (a) The function P_c has an attracting fixed point if and only if c satisfies

$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c < 3.$$

(b) The function P_c has an attracting 2-cycle if and only if c satisfies

$$|c+1| < \tfrac{1}{4}.$$

10. A **periodic region** is a maximal region $\mathcal{R}$ such that, for some positive integer p,

the function P_c has an attracting p-cycle,

for all $c \in \mathcal{R}$.

11. The function P_c has a super-attracting p-cycle if and only if

$$P_c^p(0) = 0, \text{ but } P_c^k(0) \neq 0, \text{ for } k = 1, 2, \ldots, p-1.$$

12. The number λ is a **primitive nth root of unity** if λ is a root of unity and if n is the smallest positive integer for which $\lambda^n = 1$.

13. Suppose that the function P_{c_0}, $c_0 \in \mathbb{C}$, has a p-cycle whose multiplier λ is a root of unity.

(a) **Saddle-node bifurcation at c_0** If $\lambda = 1$, then c_0 is the cusp of a cardioid-shaped periodic region $\mathcal{R}$, such that

P_c has an attracting p-cycle, for $c \in \mathcal{R}$.

(b) **Period-multiplying bifurcation at c_0** If λ is a primitive nth root of unity, for $n > 1$, then there are two periodic regions $\mathcal{R}_1$ and $\mathcal{R}_2$ whose boundaries meet at c_0 such that

$$P_c \text{ has an attracting } \begin{cases} p\text{-cycle}, & \text{for } c \in \mathcal{R}_1, \\ np\text{-cycle}, & \text{for } c \in \mathcal{R}_2. \end{cases}$$

14. The set of points which are mapped to a set E by P_c is called **the preimage set of E** and is denoted by $P_c^{-1}(E)$:

$$P_c^{-1}(E) = \{z : P_c(z) \in E\}.$$

15. A **compact disc** is a compact set whose boundary is a simple-closed smooth path.

16. If E is a compact disc and $c \notin \partial E$, then $P_c^{-1}(E)$ is

(a) one compact disc containing 0, if $c \in \mathrm{int}\, E$;

(b) two compact discs, neither containing 0, if $c \in \mathrm{ext}\, E$.

Section 5: Beyond the Mandelbrot Set

This is a reading-only section.

PART II: INDEX

angle of attack 7
Apollonian form 4

Basic Mathematical Model 6
basic quadratic functions 9
basin of attraction 9
Bernoulli's Equation 8
Blasius' Theorem 8
boundary in $\widehat{\mathbb{C}}$ 5
boundary point in $\widehat{\mathbb{C}}$ 5

circulation 6
  around an obstacle 7
compact disc 11
complex potential function 7
conjugate iteration sequences 9
conjugating function 9
critical point 7
cycle 9

degenerate streamline 6
drag 8

escape set 9
essential singularity at ∞ 3
extended complex plane 3
extended function 3
extended line 3
extended Möbius
    transformation 3

fixed point equation 9
fixed point 4, 9
  attracting 9
  indifferent 9
  repelling 9
  super-attracting 9
Flow Mapping Theorem 7
flow velocity 6
flux 6

force acting on an obstacle 8
functions
  conjugate 9

generalized circle 3
generalized open disc 5
graphical iteration 10

Implicit Formula 4
inverse points 4
inversion 4
iterate 9
iteration sequence 9

Joukowski aerofoil 7
Joukowski function 5
Julia set 10

keep set 9
Kutta-Joukowski Hypothesis 8
Kutta-Joukowski Lift
    Theorem 8

lift 8
linear function 3
locally circulation-free 6
locally flux-free 6

Mandelbrot set 10
model flow 6
Möbius transformation 3
  Implicit Formula 4
  inverse function of 4
multiplier 10

Newton–Raphson function 9
Newton–Raphson iteration
    sequence 9
normal component 6

obstacle 7
Obstacle Problem 7
open disc centred at ∞ 5

period-multiplying
    bifurcation 11
periodic point 9
  attracting 10
  indifferent 10
  repelling 10
  super-attracting 10
periodic region 11
point at infinity 3
pole of order k at ∞ 3
potential function 7
preimage set 11
primitive nth root of unity 11

reciprocal function 3
removable singularity at ∞ 3
Riemann sphere 3

saddle-node bifurcation 11
set
  completely invariant 9
  connected 10
  disconnected 10
sink 6
source 6
stagnation point 6
standard triple of points 4
stereographic projection 3
stream function 7
streamline 6

tangential component 6
trailing edge 7

uniform flow 6
unit-speed parametrization 6

vortex 6

Printed in the United Kingdom by
The Open University